7 Post-Minkowskian theory: Implementation

The theory was formulated in Chapter 6, and now we must get our hands dirty with its implementation. In this chapter we construct the *second post-Minkowskian approximation* to the metric of a curved spacetime produced by a bounded distribution of matter. For concreteness we choose the matter to consist of a perfect fluid. Our treatment allows the fluid to be of one piece (in the case of a single body), or broken down into a number of disconnected components (in the case of an N-body system).

Although the post-Minkowskian approximation does not *require* slow motion, we shall nevertheless assume that the fluid is subjected to a slow-motion condition of the sort described in Sec. 6.3.2: if v_c is a characteristic velocity within the fluid, we insist that $v_c/c \ll 1$. This amounts to incorporating a *post-Newtonian expansion* within the post-Minkowskian approximation. We do this for two reasons. First, our ultimate goal is to describe situations of astrophysical interest, and the virial theorem implies that $U \sim v^2$ for any gravitationally bound system; weak fields are naturally accompanied by slow motion. Second, any attempt to keep the velocities arbitrary in the post-Minkowskian expansion quickly leads to calculations that are unmanageable, and we prefer to avoid these complications here.

We begin in Sec. 7.1 by assembling the required tools and exploring the general structure of the gravitational potentials in the near and wave zones. In Sec. 7.2 we perform the first iteration of the relaxed field equations, and the outcome of this calculation is used as input in the second iteration, carried out in Sec. 7.3 for the near zone, and in Sec. 7.4 for the wave zone. Our main results are summarized in Boxes 7.5 and 7.7.

Before we proceed it is perhaps useful to recall the main results of the preceding chapter. We saw that in the Landau–Lifshitz formulation of general relativity, the Einstein field equations take the form of a wave equation for the gravitational potentials $h^{\alpha\beta} := \eta^{\alpha\beta} - \sqrt{-g}g^{\alpha\beta}$, together with the harmonic-gauge condition $\partial_\beta h^{\alpha\beta} = 0$; this is formally equivalent to the conservation equation $\partial_\beta \tau^{\alpha\beta} = 0$ for the effective energy-momentum pseudotensor, which acts as the source term in the wave equation. Each post-Minkowskian iteration of the wave equation gives rise to a new expression for the source, which is inserted back into the wave equation for the next iteration. After each iteration $h^{\alpha\beta}$ is expressed as an integral of the source over the past light cone of the field point $(t, \boldsymbol{x})$. Because the support of $\tau^{\alpha\beta}$ is not limited to the matter distribution, the domain of integration covers the entire light cone, and it is decomposed into a near-zone domain $\mathscr{N}$ and a wave-zone domain $\mathscr{W}$; the gravitational potentials are expressed as $h^{\alpha\beta} = h^{\alpha\beta}_{\mathscr{N}} + h^{\alpha\beta}_{\mathscr{W}}$. The boundary between the near and wave zones is placed at an arbitrary radius $r = \mathcal{R}$, with $\mathcal{R}$ chosen to be of the same order of magnitude as a characteristic wavelength of the

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gravitational radiation; while $h^{\alpha\beta}_{\mathscr{N}}$ and $h^{\alpha\beta}_{\mathscr{W}}$ individually depend on $\mathcal{R}$, their sum is guaranteed to be independent of the cutoff radius, and this dependence can therefore be ignored.

And now onward with an explicit implementation of these ideas.

7.1 Assembling the tools

We begin by gathering the various tools, formulae, and assumptions that are required in implementation of the post-Minkowskian expansion. Our discussion here will set the stage for the various applications to come, to post-Newtonian theory (Chapters 8 to 10), to gravitational waves (Chapter 11), and to gravitational radiation reaction (Chapter 12).

7.1.1 Fluid variables

A description of the laws of fluid mechanics in curved spacetime was presented in Sec. 5.3. There we saw that the matter variables m that are relevant to a perfect fluid are the proper mass density ρ, the proper internal energy density ϵ, the pressure p, and the velocity field u^α. The energy-momentum tensor of a perfect fluid is

$$T^{\alpha\beta} = (\rho + \epsilon/c^2 + p/c^2)u^\alpha u^\beta + pg^{\alpha\beta} . \tag{7.1}$$

The fluid dynamics is subjected to two conservation statements, a conservation of rest-mass expressed by $\nabla_\alpha(\rho u^\alpha) = 0$, and a conservation of energy-momentum expressed by $\nabla_\beta T^{\alpha\beta} = 0$.

For our purposes it is convenient to employ a slightly different set of matter variables. Noting that the components of u^α are not all independent (because of the normalization condition $g_{\alpha\beta}u^\alpha u^\beta = -c^2$), we express the four-velocity field as

$$u^\alpha = \gamma(c, \boldsymbol{v}) , \tag{7.2}$$

in terms of a three-velocity field $\boldsymbol{v}$ and a factor $\gamma := u^0/c$ that can be determined in terms of $\boldsymbol{v}$ by the normalization condition. Making the substitution within the equation of mass conservation, we find that it can be expressed in the form

$$\partial_t \rho^* + \partial_j(\rho^* v^j) = 0 , \tag{7.3}$$

in terms of a rescaled mass density defined by

$$\rho^* := \sqrt{-g}\gamma\rho = \sqrt{-g}\,\rho\, u^0/c . \tag{7.4}$$

To arrive at Eq. (7.3) we made use of the divergence identity of Eq. (5.40). Finally, we shall use $\Pi := \epsilon/\rho$ instead of ϵ; this is the fluid's internal energy per unit mass. Our final set of matter variables is therefore

$$\mathsf{m} := \{\rho^*, p, \Pi, \boldsymbol{v}\} , \tag{7.5}$$

and all other fluid variables can be determined in terms of this set.

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The continuity equation (7.3) plays an important role in the description of a perfect fluid. We observe that unlike $\nabla_\beta T^{\alpha\beta} = 0$, which constrains the dynamics of the fluid, the statement of mass conservation is entirely kinematical in nature. Equation (7.3) states that the rest-mass of a fluid element does not change as we follow its motion within the fluid; this is tantamount to defining what one means by the phrase "fluid element," and the statement is indeed a piece of the kinematical description of the fluid. This is quite distinct, for example, from the statement of the first law of thermodynamics, $d\Pi - (p/\rho^2)\,d\rho = 0$ (refer to Sec. 1.4.2), which is dynamical in nature.

We assume that the fluid is subjected to a slow-motion condition. Recalling the scaling quantities introduced in Sec. 6.3.2, we have that r_c is the radius of a sphere that surrounds the matter distribution, t_c is a characteristic time scale associated with the fluid motions, $v_c = r_c/t_c$ is a characteristic velocity within the fluid, $\lambda_c = ct_c$ is a characteristic wavelength of the gravitational radiation produced by the moving fluid, and m_c is the characteristic mass of the matter distribution. We demand that $v_c/c \ll 1$, which implies that the fluid is situated deep within the near zone: $r_c \ll \lambda_c$.

The slow-motion condition gives rise to a hierarchy between the components of the energy-momentum tensor. From Eq. (7.1) we have the approximate relations $T^{00} \simeq \rho^* c^2$, $T^{0j} \simeq \rho^* v^j c$, and $T^{jk} \simeq \rho^* v^j v^k + p\,\delta^{jk}$, and these imply

$$T^{0j}/T^{00} \sim v_c/c\,, \qquad T^{jk}/T^{00} \sim (v_c/c)^2\,. \tag{7.6}$$

A glance at Eq. (6.51) then reveals that this hierarchy is inherited by the gravitational potentials:

$$h^{0j}/h^{00} \sim v_c/c\,, \qquad h^{jk}/h^{00} \sim (v_c/c)^2\,. \tag{7.7}$$

It is useful to express these relations more directly as

$$T^{00} = O(c^2)\,, \qquad T^{0j} = O(c)\,, \qquad T^{jk} = O(1)\,, \tag{7.8}$$

and (taking into account the factor c^{-4} in the field equations)

$$h^{00} = O(c^{-2}), \qquad h^{0j} = O(c^{-3}), \qquad h^{jk} = O(c^{-4}), \tag{7.9}$$

thereby introducing c^{-2} as a *post-Newtonian expansion parameter*. This notation serves as a powerful mnemonic to judge the importance of various terms in a post-Newtonian expansion. But it is a notational shortcut that must be used with care; it should be remembered, for example, that a relation such as $T^{jk} = O(1)$ really stands for something more meaningful, such as $T^{jk}/T^{00} \sim (v_c/c)^2$.

7.1.2 General structure of the potentials: Near zone

Having introduced the matter variables, the slow-motion condition, and the post-Newtonian hierarchy, we turn next to an examination of the general structure of the gravitational potentials $h^{\alpha\beta}$. These are determined by the relaxed field equations

$$\Box h^{\alpha\beta} = -\frac{16\pi G}{c^4}\tau^{\alpha\beta}, \tag{7.10}$$

in which

$$\tau^{\alpha\beta} = (-g)\big(T^{\alpha\beta} + t^{\alpha\beta}_{\mathrm{LL}} + t^{\alpha\beta}_{\mathrm{H}}\big) \tag{7.11}$$

is the effective energy-momentum pseudotensor of Eq. (6.52). We decompose the potentials as $h^{\alpha\beta} = h^{\alpha\beta}_{\mathscr{N}} + h^{\alpha\beta}_{\mathscr{W}}$, and first examine them when the field point $\boldsymbol{x}$ is in the near zone, where $r := |\boldsymbol{x}| < \mathcal{R}$.

Consulting Box 6.7, we see that $h^{\alpha\beta}_{\mathscr{N}}$ can be expressed as the expansion

$$h^{\alpha\beta}_{\mathscr{N}}(t,\boldsymbol{x}) = \frac{4G}{c^4}\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!c^\ell}\left(\frac{\partial}{\partial t}\right)^\ell \int_{\mathscr{M}} \tau^{\alpha\beta}(t,\boldsymbol{x}')|\boldsymbol{x}-\boldsymbol{x}'|^{\ell-1}\,d^3x' , \tag{7.12}$$

in which $\mathscr{M}$ is a surface of constant time bounded externally by $r' := |\boldsymbol{x}'| = \mathcal{R}$. The first few terms are

$$\begin{aligned} h^{\alpha\beta}_{\mathscr{N}}(t,\boldsymbol{x}) = \frac{4G}{c^4}\bigg[&\int_{\mathscr{M}}\frac{\tau^{\alpha\beta}(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x' - \frac{1}{c}\frac{d}{dt}\int_{\mathscr{M}}\tau^{\alpha\beta}(t,\boldsymbol{x}')\,d^3x' \\ &+ \frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\int_{\mathscr{M}}\tau^{\alpha\beta}(t,\boldsymbol{x}')|\boldsymbol{x}-\boldsymbol{x}'|\,d^3x' \\ &- \frac{1}{6c^3}\frac{\partial^3}{\partial t^3}\int_{\mathscr{M}}\tau^{\alpha\beta}(t,\boldsymbol{x}')(r^2 - 2\boldsymbol{x}\cdot\boldsymbol{x}' + r'^2)\,d^3x' + \cdots\bigg], \end{aligned} \tag{7.13}$$

and we see that each successive term comes with an additional factor of c^{-1}, signifying that it is smaller than the previous term by a factor of order $v_c/c \ll 1$. This is our first encounter with a post-Newtonian expansion in powers of c^{-2}, with fractional orders assigned to odd powers of c^{-1}.

The expansion of Eq. (7.13) is a direct consequence of the relaxed field equations. It simplifies when we take into account the conservation statement $\partial_\beta\tau^{\alpha\beta} = 0$ for the energy-momentum pseudotensor. When we examine the expansion for $h^{00}_{\mathscr{N}}$, for example, we note that the second term is given by $-\int_{\mathscr{M}}\partial_0\tau^{00}\,d^3x'$. The conservation statement allows us to make the substitution $\partial_0\tau^{00} = -\partial_j\tau^{0j}$ inside the integral, which can then, by Gauss's theorem, be converted to a surface integral over $\partial\mathscr{M}$, the boundary of the region $\mathscr{M}$; this is the surface $r' = \mathcal{R}$. The surface integral would vanish if τ^{0j} were confined to the near zone, and in this case the expansion for $h^{00}_{\mathscr{N}}$ would skip the term at order c^{-1}. In general, however, τ^{0j} extends beyond the near zone, and the surface integral does not vanish. But since τ^{0j} is constructed from the potentials, the surface integral can be estimated and shown to be of a very high order in the post-Newtonian expansion, well beyond any order that we will encounter in this book. In practice, therefore, we can appeal to energy conservation and eliminate the second term in the expansion for $h^{00}_{\mathscr{N}}$.

In fact, the conservation equations $\partial_\beta\tau^{\alpha\beta} = 0$ can be put to good use to simplify and organize many terms in the expansion of $h^{\alpha\beta}_{\mathscr{N}}$. Particularly useful are a number of identities that follow from the conservation equations, namely,

$$\tau^{0j} = \partial_0\big(\tau^{00}x^j\big) + \partial_k\big(\tau^{0k}x^j\big), \tag{7.14a}$$

$$\tau^{jk} = \frac{1}{2}\partial_{00}\big(\tau^{00}x^jx^k\big) + \frac{1}{2}\partial_p\big(2\tau^{p(j}x^{k)} - \partial_q\tau^{pq}x^jx^k\big), \tag{7.14b}$$

(*continued overleaf*)

$$\tau^{0j}x^k = \frac{1}{2}\partial_0\big(\tau^{00}x^jx^k\big) + \tau^{0[j}x^{k]} + \frac{1}{2}\partial_p\big(\tau^{0p}x^jx^k\big)\,, \tag{7.14c}$$

$$\begin{aligned}\tau^{jk}x^n &= \frac{1}{2}\partial_0\big(2\tau^{0(j}x^{k)}x^n - \tau^{0n}x^jx^k\big)\\ &\quad + \frac{1}{2}\partial_p\big(2\tau^{p(j}x^{k)}x^n - \tau^{np}x^jx^k\big)\,,\end{aligned} \tag{7.14d}$$

in which round and square brackets surrounding indices denote symmetrized and anti-symmetrized combinations, respectively. Exploiting these identities, we find after some manipulations that the various components of the gravitational potentials are now given by

$$\begin{aligned}h^{00}_{\mathscr{N}} = \frac{4G}{c^2}\bigg\{&\int_{\mathscr{M}} \frac{c^{-2}\tau^{00}}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x' + \frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\int_{\mathscr{M}} c^{-2}\tau^{00}|\boldsymbol{x}-\boldsymbol{x}'|\,d^3x'\\ &- \frac{1}{6c^3}\overset{(3)}{\mathcal{I}}{}^{kk}(t) + \frac{1}{24c^4}\frac{\partial^4}{\partial t^4}\int_{\mathscr{M}} c^{-2}\tau^{00}|\boldsymbol{x}-\boldsymbol{x}'|^3\,d^3x'\\ &- \frac{1}{120c^5}\Big[(4x^kx^l + 2r^2\delta^{kl})\overset{(5)}{\mathcal{I}}{}^{kl}(t) - 4x^k\overset{(5)}{\mathcal{I}}{}^{kll}(t) + \overset{(5)}{\mathcal{I}}{}^{kkll}(t)\Big]\\ &+ O(c^{-6})\bigg\} + h^{00}[\partial\mathscr{M}]\,,\end{aligned} \tag{7.15a}$$

$$\begin{aligned}h^{0j}_{\mathscr{N}} = \frac{4G}{c^3}\bigg\{&\int_{\mathscr{M}} \frac{c^{-1}\tau^{0j}}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x' + \frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\int_{\mathscr{M}} c^{-1}\tau^{0j}|\boldsymbol{x}-\boldsymbol{x}'|\,d^3x'\\ &+ \frac{1}{18c^3}\Big[3x^k\overset{(4)}{\mathcal{I}}{}^{jk}(t) - \overset{(4)}{\mathcal{I}}{}^{jkk}(t) + 2\epsilon^{mjk}\overset{(3)}{\mathcal{J}}{}^{mk}(t)\Big]\\ &+ O(c^{-4})\bigg\} + h^{0j}[\partial\mathscr{M}]\,,\end{aligned} \tag{7.15b}$$

$$\begin{aligned}h^{jk}_{\mathscr{N}} = \frac{4G}{c^4}\bigg\{&\int_{\mathscr{M}} \frac{\tau^{jk}}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x' - \frac{1}{2c}\overset{(3)}{\mathcal{I}}{}^{jk}(t) + \frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\int_{\mathscr{M}} \tau^{jk}|\boldsymbol{x}-\boldsymbol{x}'|\,d^3x'\\ &- \frac{1}{36c^3}\Big[3r^2\overset{(5)}{\mathcal{I}}{}^{jk}(t) - 2x^m\overset{(5)}{\mathcal{I}}{}^{jkm}(t) - 8x^n\epsilon^{mn(j}\overset{(4)}{\mathcal{J}}{}^{m|k)}(t) + 6\overset{(3)}{\mathcal{M}}{}^{jkmm}(t)\Big]\\ &+ O(c^{-4})\bigg\} + h^{jk}[\partial\mathscr{M}]\,,\end{aligned} \tag{7.15c}$$

in which $\tau^{\alpha\beta}$ is expressed as a function of t and $\boldsymbol{x}'$ inside the integrals, a number within brackets placed above a symbol such as $\mathcal{I}^{jk}$ indicates the number of differentiations with respect to time, and $h^{\alpha\beta}[\partial\mathscr{M}]$ denotes the collected surface terms generated during our manipulations of the integrals (the details will not be displayed here). We have also introduced the following notation for the multipole moments of the source $\tau^{\alpha\beta}$:

$$\mathcal{I}^L(t) := \int_{\mathscr{M}} c^{-2}\tau^{00}(t,\boldsymbol{x})x^L\,d^3x\,, \tag{7.16a}$$

$$\mathcal{J}^{jL}(t) := \epsilon^{jab}\int_{\mathscr{M}} c^{-1}\tau^{0b}(t,\boldsymbol{x})x^{aL}\,d^3x\,, \tag{7.16b}$$

$$\mathcal{M}^{jkL} := \int_{\mathscr{M}} \tau^{jk}(t,\boldsymbol{x})x^L\,d^3x\,, \tag{7.16c}$$

in which L is a multi-index containing a number ℓ of individual indices, so that $A^L := A^{j_1 j_2 \cdots j_\ell}$ and $x^L := x^{j_1} x^{j_2} \ldots x^{j_\ell}$.

There is a lot to take in with the expansions of Eq. (7.15), and we shall now take the time to describe the structure of $h^{00}_{\mathcal{N}}$ in some detail. We begin with the first term on the right-hand side of Eq. (7.15a), and observe that it leads off at order c^{-2} with a Newtonian-like potential associated with the mass density $c^{-2}\tau^{00} \sim \rho^*$. Embedded within this term are corrections of order $(v_c/c)^2$ and higher that enter the detailed expression for τ^{00}, as well as corrections of order G and higher that arise in previous iterations of the relaxed field equations. But the leading contribution gives rise to Newtonian gravity.

The integral that appears in the second term in $h^{00}_{\mathcal{N}}$ is known as a superpotential, because the factor $|\boldsymbol{x} - \boldsymbol{x}'|$ appears in the numerator instead of the denominator; as we shall see, a superpotential is a potential sourced by another potential. Because of the time derivatives, this term leads off at order c^{-2} relative to the Newtonian term, or at overall order c^{-4} in $h^{00}_{\mathcal{N}}$; it is a "first post-Newtonian correction," or 1PN correction, to the gravitational potential. It also contains higher-order corrections coming from higher-order terms in $c^{-2}\tau^{00}$, just as we saw previously for the leading-order, Newtonian term. It is instructive to note that the superpotential itself is of order $m_c r$, but since each time derivative produces a factor of $t_c^{-1} = v_c/r_c$, its contribution to $h^{00}_{\mathcal{N}}$ is a factor of order $(v_c/c)^2$ smaller than the Newtonian potential when r is comparable to r_c.

The third term in $h^{00}_{\mathcal{N}}$ involves three time derivatives of $\mathcal{I}^{kk}(t)$, the trace of the mass quadrupole moment. The factor of c^{-3} in front indicates that this term is a factor of order $(v_c/c)^3$ smaller than the leading, Newtonian term, and therefore represents a 1.5PN contribution to the gravitational potential. We will show below that since this term depends on t only, it can always be absorbed into a redefinition of the time coordinate, and therefore be removed by a coordinate transformation. This observation suggests that the 1.5PN term does not play a physical role, and we shall have occasion to show that such is indeed the case. The expression for the 1.5PN term displayed in Eq. (7.15a) is derived in Box 7.1.

The integral that appears in the fourth term in $h^{00}_{\mathcal{N}}$ is sometimes called a superduperpotential because of the presence of $|\boldsymbol{x} - \boldsymbol{x}'|^3$ in the numerator; a superduperpotential is a potential sourced by a superpotential. Because of the time derivatives, this term leads off at order c^{-4} relative to the Newtonian term, and therefore represents a 2PN correction to the gravitational potential.

We now examine the fifth set of terms. The first member of the set involves the mass quadrupole moment differentiated five times with respect to time, and it scales as

$$\frac{r_c^2}{c^5}\frac{1}{t_c^5} m_c r_c^2 = (v_c/c)^5 \frac{m_c}{r_c}, \tag{7.17}$$

which is a correction of order $(v_c/c)^5$ relative to the Newtonian term. The other members have the same scaling, and this group of terms give rise to a 2.5PN correction to the gravitational potential. Unlike the 1.5PN term, this group depends on the spatial coordinates in addition to time, and it cannot be removed by a coordinate transformation. It gives rise to real, physical effects on the system. The nature of these effects can be anticipated from the fact that the 2.5PN terms involve an odd number of time derivatives, and are therefore antisymmetric under a time reflection $t \to -t$; this is in contrast with the 1PN and 2PN terms, which are symmetric under the time reflection. This property is associated with dissipative processes taking place within the system, representing a radiative loss of energy to gravitational waves. The 2.5PN contributions to the gravitational potentials are known as *radiation-reaction potentials*, and their effects will be explored in detail in Chapter 12.

Turning next to the other components of the gravitational potentials, we observe that they have a very similar structure. The component $h^{0j}_{\mathscr{N}}$ leads off at order c^{-3} with a Newtonian-like potential sourced by the mass-current density $c^{-1}\tau^{0j} \sim \rho^* v^j$. Comparing this with the leading term in $h^{00}_{\mathscr{N}}$, we see that it is smaller by a factor of order v_c/c, and it would be tempting to assign a 0.5PN label to this term. As we shall see below, however, all effects arising from $h^{0j}_{\mathscr{N}}$ will be the result of a coupling with other quantities that also scale as v_c/c; the result is a 1PN correction to the leading, Newtonian effect. Keeping this context in mind, it is appropriate to reset the post-Newtonian counter and to declare that the leading term in $h^{0j}_{\mathscr{N}}$ makes a 1PN contribution to the gravitational potentials. The expansion of $h^{0j}_{\mathscr{N}}$ continues with a superpotential term at order c^{-5} which is assigned a 2PN label, and this is followed by 2.5PN contributions. The absence of a term at order c^{-4} is a consequence of momentum conservation; the manipulations that led to the disappearance of the c^{-3} term in $h^{00}_{\mathscr{N}}$ lead to the same conclusion here, and in both cases we see that these terms are absorbed in the surface integrals $h^{\alpha\beta}[\partial\mathscr{M}]$.

And finally, the components $h^{jk}_{\mathscr{N}}$ lead off at order c^{-4} with a Newtonian-like potential sourced by $\tau^{jk} \sim \rho^* v^j v^k$; this is smaller than the leading term in $h^{00}_{\mathscr{N}}$ by a factor of order $(v_c/c)^2$ and represents a 1PN contribution to the gravitational potentials. The next term, involving a single time derivative, does *not* vanish; use of Eqs. (7.14c) and (7.14d) converts it to three time derivatives of the mass quadrupole moment. This term represents a 1.5PN contribution, and it is followed by a superpotential term at 2PN order, and a set of 2.5PN contributions.

The potentials $h^{\alpha\beta}_{\mathscr{N}}$ provide the near-zone portion of the light-cone integral giving $h^{\alpha\beta}$ in terms of $\tau^{\alpha\beta}$, and we have yet to examine the wave-zone portion $h^{\alpha\beta}_{\mathscr{W}}$. We recall from Box. 6.7 that this can be expressed as

$$h^{\alpha\beta}_{\mathscr{W}}(t,\boldsymbol{x}) = \frac{4G}{c^4}\frac{n^{\langle L\rangle}}{r}\left\{\int_{\mathcal{R}-r}^{\mathcal{R}} ds f^{\alpha\beta}(\tau - 2s/c)A(s,r) + \int_{\mathcal{R}}^{\infty} ds f^{\alpha\beta}(\tau - 2s/c)B(s,r)\right\}, \tag{7.18}$$

when $\tau^{\alpha\beta}$ can be put in the specific form

$$\tau^{\alpha\beta} = \frac{1}{4\pi}\frac{f^{\alpha\beta}(\tau)}{r^n}n^{\langle L\rangle}. \tag{7.19}$$

Here $\tau = t - r/c$ is retarded time, $n^{\langle L\rangle}$ is an angular STF tensor of the sort introduced back in Sec. 1.5.3, and the functions $A(s,r)$ and $B(s,r)$ are defined by Eq. (6.104). Although restrictive, the expression of Eq. (7.18) is nevertheless useful because the wave-zone sources

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$\tau^{\alpha\beta}$ encountered below will always be decomposed in the elementary forms displayed in Eq. (7.19); the complete $h^{\alpha\beta}_{\mathscr{W}}$ can then be obtained by summing over these elementary contributions.

Little more can be said about the general structure of $h^{\alpha\beta}_{\mathscr{W}}$ in the near zone. The sources $f^{\alpha\beta}$ vanish in the first iteration of the relaxed field equations, because we are instructed to set $h^{\alpha\beta} = 0$ in $\tau^{\alpha\beta}$ and the material source is confined to the near zone. In the second and higher iterations, $h^{\alpha\beta}$ is no longer zero, and $\tau^{\alpha\beta}$ now extends into the wave zone; in these cases we have no choice but to plow through the detailed calculations to see what contribution $h^{\alpha\beta}_{\mathscr{W}}$ might make. We encounter some of these calculations later in this chapter, and then again in Chapter 11.

Box 7.1 Radiation-reaction terms in the potentials

To illustrate how the various radiation-reaction terms arise in the potentials, we examine the contribution

$$-\frac{1}{6c^3}\left(\frac{\partial}{\partial t}\right)^3 \int_{\mathscr{M}} \tau^{00}(t,\boldsymbol{x}')(r^2 - 2\boldsymbol{x}\cdot\boldsymbol{x}' + r'^2)\, d^3x'$$

to $h^{00}_{\mathscr{N}}$; this is the third line in Eq. (7.13). In the first term, r^2 can be brought outside the integral, giving $-\frac{1}{6}c^{-2}r^2\partial_t^2 \int_{\mathscr{M}} \partial_0\tau^{00}\, d^3x' = \frac{1}{6}c^{-2}r^2\partial_t^2 \int_{\mathscr{M}} \partial_j\tau^{0j}\, d^3x'$, which becomes a surface term, reflecting the fact that energy is conserved apart from a tiny flux of gravitational radiation. In the second term, $\boldsymbol{x}$ can be brought outside the integral, giving $\frac{1}{3}c^{-2}x^j\partial_t^2 \int_{\mathscr{M}} \partial_0\tau^{00}x'^j\, d^3x' = \frac{1}{3}c^{-2}x^j\partial_t^2 \int_{\mathscr{M}} \tau^{0j}\, d^3x'$ plus a surface term. This yields $-\frac{1}{3}c^{-1}x^j\partial_t \int_{\mathscr{M}} \partial_k\tau^{kj}\, d^3x'$, which gives another surface term. The elimination of this term reflects the conservation of momentum. The third term survives, giving $-\frac{1}{6}c^{-3}\dddot{\mathcal{I}}^{kk}$ as shown in Eq. (7.15a).

The next term in $h^{00}_{\mathscr{N}}$ involving an odd number of time derivatives is

$$-\frac{1}{120c^5}\left(\frac{\partial}{\partial t}\right)^5 \int_{\mathscr{M}} \tau^{00}(t,\boldsymbol{x}')\Big[\, r^4 - 4r^2\boldsymbol{x}\cdot\boldsymbol{x}' + 4(\boldsymbol{x}\cdot\boldsymbol{x}')^2 + 2r^2r'^2 - 4r'^2\boldsymbol{x}\cdot\boldsymbol{x}' + r'^4\Big]\, d^3x' .$$

The first two terms can be shown to become surface integrals by appealing to the conservation identities of Eqs. (7.14), and the remaining four terms are displayed in Eq. (7.15a). Similar manipulations, albeit becoming progressively more complicated, yield the corresponding terms displayed in Eqs. (7.15) for $h^{0j}_{\mathscr{N}}$ and $h^{jk}_{\mathscr{N}}$.

7.1.3 Near-zone metric

We will need to construct the spacetime metric $g_{\alpha\beta}$ from the gravitational potentials $h^{\alpha\beta}$. The link is provided by the gothic inverse metric $\mathfrak{g}^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta}$, which is related to the inverse metric $g^{\alpha\beta}$ by $\mathfrak{g}^{\alpha\beta} = \sqrt{-g}g^{\alpha\beta}$. The inverse relation is $g_{\alpha\beta} = \sqrt{-\mathfrak{g}}\,\mathfrak{g}_{\alpha\beta}$, in which $\mathfrak{g}_{\alpha\beta}$ is the matrix inverse to $\mathfrak{g}^{\alpha\beta}$, and $\mathfrak{g} := \det[\mathfrak{g}^{\alpha\beta}]$. Given that $h^{\alpha\beta}$ is of order G, we can solve these equations and obtain the metric and its inverse as post-Minkowskian expansions in powers of G, and express the results in terms of the potentials $h^{\alpha\beta}$. We find

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} - \frac{1}{2}h\eta_{\alpha\beta} + h_{\alpha\mu}h^{\mu}_{\ \beta} - \frac{1}{2}hh_{\alpha\beta} + \left(\frac{1}{8}h^2 - \frac{1}{4}h^{\mu\nu}h_{\mu\nu}\right)\eta_{\alpha\beta} + O(G^3), \tag{7.20a}$$

$$g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta} + \frac{1}{2}h\eta^{\alpha\beta} - \frac{1}{2}hh^{\alpha\beta} + \left(\frac{1}{8}h^2 + \frac{1}{4}h^{\mu\nu}h_{\mu\nu}\right)\eta^{\alpha\beta} + O(G^3), \tag{7.20b}$$

$$(-g) = 1 - h + \frac{1}{2}h^2 - \frac{1}{2}h^{\mu\nu}h_{\mu\nu} + O(G^3), \tag{7.20c}$$

$$\sqrt{-g} = 1 - \frac{1}{2}h + \frac{1}{8}h^2 - \frac{1}{4}h^{\mu\nu}h_{\mu\nu} + O(G^3). \tag{7.20d}$$

It is understood that here, indices on $h^{\alpha\beta}$ are lowered with the Minkowski metric, so that $h_{\alpha\beta} := \eta_{\alpha\mu}\eta_{\beta\nu}h^{\mu\nu}$ and $h := \eta_{\mu\nu}h^{\mu\nu}$.

In practice, the construction of the metric from the potentials depends on the context, which dictates the degree of accuracy required of each metric component. Suppose that we are specifically interested in determining the geodesic motion of a slowly-moving particle in the near zone of a weakly-curved spacetime. As we saw back in Sec. 5.2.3, the motion is governed by a Lagrangian L given by

$$\begin{aligned} L &= -mc\sqrt{-g_{\alpha\beta}\frac{dr^\alpha}{dt}\frac{dr^\beta}{dt}} \\ &= -mc^2\sqrt{-g_{00} - 2g_{0j}v^j/c - g_{jk}v^jv^k/c^2}, \end{aligned} \tag{7.21}$$

where $r^\alpha = (ct, \boldsymbol{r})$ describes the particle's position in spacetime, and $v^j = dr^j/dt$ is its three-dimensional velocity vector. Newtonian gravity is reproduced by inserting the approximations $g_{00} = -1 + 2U/c^2 + O(c^{-4})$, $g_{0j} = O(c^{-3})$, and $g_{jk} = \delta_{jk} + O(c^{-2})$ within the Lagrangian, and expanding the square root to order c^{-2}; this yields

$$L = -mc^2 + \frac{1}{2}mv^2 + mU + O(c^{-2}), \tag{7.22}$$

in which U is the Newtonian potential. The first term is an irrelevant constant, and we indeed recognize $\frac{1}{2}mv^2 + mU$ as the Lagrangian of Newtonian gravity; the remaining terms of order c^{-2} are 1PN corrections. This simple exercise teaches us that a contribution of order c^{-2} to g_{00} is a Newtonian term, but that a term of order c^{-2} in g_{jk} is actually a post-Newtonian correction.

If we now want the post-Newtonian corrections to the motion, we must evaluate the Lagrangian to order c^{-2}, and this requires calculation of the metric to the following orders of approximation:

$$\begin{aligned} &O(c^{-4}) \quad \text{for} \quad g_{00}\,, \\ &O(c^{-3}) \quad \text{for} \quad g_{0j}\,, \\ &O(c^{-2}) \quad \text{for} \quad g_{jk}\,. \end{aligned}$$

In this case, a term of order c^{-4} in g_{00} gives rise to a post-Newtonian correction to the Lagrangian. The same is true of a term of order c^{-3} in g_{0j}, because it multiplies v^j/c in the Lagrangian, making the combination a term of order c^{-4}. And the same is also true of a term of order c^{-2} in g_{jk}, because it multiplies $v^j v^k/c^2$ in the Lagrangian. Generalizing the argument, we find that determination of the motion to nPN order requires calculation of the metric to the orders

$$\begin{aligned} O(c^{-2n-2}) &\quad \text{for} \quad g_{00}\,, \\ O(c^{-2n-1}) &\quad \text{for} \quad g_{0j}\,, \\ O(c^{-2n}) &\quad \text{for} \quad g_{jk}\,; \end{aligned}$$

as usual the orders in c^{-1} descend because of the additional factors of v^j/c in the Lagrangian.

Suppose next that we wish to determine the motion of a test body to 2.5PN order. The previous discussion indicates that we need g_{00} to order c^{-7}, g_{0j} to order c^{-6}, and g_{jk} to order c^{-5}. The metric is obtained from the potentials $h^{\alpha\beta}$, and recalling from Eqs. (7.15) that h^{00} leads off at order c^{-2}, h^{0j} at order c^{-3}, and h^{jk} at order c^{-4}, we find from Eq. (7.20) that the appropriate expression is

$$\begin{aligned} g_{00} &= -1 + \frac{1}{2}h^{00} - \frac{3}{8}\left(h^{00}\right)^2 + \frac{5}{16}\left(h^{00}\right)^3 + \frac{1}{2}h^{kk}\left(1 - \frac{1}{2}h^{00}\right) + \frac{1}{2}h^{0j}h^{0j} \\ &\quad + O(c^{-8})\,, \end{aligned} \tag{7.23a}$$

$$g_{0j} = -h^{0j}\left(1 - \frac{1}{2}h^{00}\right) + O(c^{-7})\,, \tag{7.23b}$$

$$g_{jk} = \delta_{jk}\left[1 + \frac{1}{2}h^{00} - \frac{1}{8}\left(h^{00}\right)^2\right] + h^{jk} - \frac{1}{2}\delta_{jk}h^{mm} + O(c^{-6})\,, \tag{7.23c}$$

$$(-g) = 1 + h^{00} - h^{kk} + O(c^{-6})\,. \tag{7.23d}$$

To arrive at these results we actually had to carry the expansion of Eq. (7.20) to the third order in G, in order to capture the $(h^{00})^3$ term in g_{00}; this term is of order c^{-6}, and it is required for a complete expansion accurate through 2.5PN order.

Examining Eqs. (7.23), we begin to see how different orders in the post-Newtonian expansion of $h^{\alpha\beta}$ contribute to the metric. Beginning with g_{00}, we see from Eq. (7.15) that h^{00} contributes at all orders, from Newtonian order (c^{-2}) through 2.5PN order (c^{-7}), that h^{0j} contributes at 2PN order (c^{-6}) only, and that h^{jk} contributes at all orders beyond the Newtonian order (c^{-4}, c^{-5}, c^{-6}, and c^{-7}). With g_{0j} we find that h^{00} contributes at 2PN order (c^{-5}) only, while h^{0j} contributes at 1PN, 2PN, and 2.5PN orders (c^{-3}, c^{-5}, and c^{-6}). And finally, with g_{jk} we see that h^{00} contributes at 1PN, 2PN, and 2.5PN orders (c^{-2}, c^{-4}, and c^{-5}), while h^{jk} contributes at 2PN and 2.5PN orders (c^{-4} and c^{-5}).

We observe that each power of c^{-2} assigned to a contribution to $g_{\alpha\beta}$ translates to a specific post-Newtonian order. The translation, however, depends on the context. When the metric is examined in isolation, a term of order c^{-2} in g_{jk} could be declared to be of the same post-Newtonian order as a term of order c^{-2} in g_{00}. But when the metric is examined in the context of determining the motion of a slowly-moving particle, the c^{-2} term in g_{jk} is appropriately declared to be a 1PN term, while the c^{-2} term in g_{00} is labeled as a Newtonian contribution. The translation is again different when the motion is highly relativistic, with velocities v^j approaching the speed of light. In this case the coupling of the metric with powers of $v^j/c \simeq 1$ does not alter the post-Newtonian order, and a c^{-2} term in g_{jk} would again be declared to be a Newtonian contribution. Context is everything, and it must be specified before a meaningful post-Newtonian order can be assigned to a given expression.

Our considerations in this chapter, and the three chapters that follow, will be limited to post-Newtonian gravity, in which corrections of 2PN order and higher are neglected. In this 1PN context our expansion for the metric can be truncated to

$$g_{00} = -1 + \frac{1}{2}h^{00} - \frac{3}{8}(h^{00})^2 + \frac{1}{2}h^{kk} + O(c^{-6}), \tag{7.24a}$$

$$g_{0j} = -h^{0j} + O(c^{-5}), \tag{7.24b}$$

$$g_{jk} = \delta_{jk}\left(1 + \frac{1}{2}h^{00}\right) + O(c^{-4}), \tag{7.24c}$$

$$(-g) = 1 + h^{00} + O(c^{-4}). \tag{7.24d}$$

We return to the higher-order corrections in Chapter 12, when we study the effects of gravitational reaction in the near zone. There we shall be interested in all 2.5PN terms in the metric, those that scale as c^{-7} in g_{00}, as c^{-6} in g_{0j}, and as c^{-5} in g_{jk}. We shall see that with suitable care, we can study these radiative effects independently of the 1PN or 2PN influences.

7.1.4 General structure of the potentials: Wave zone

We proceed with an examination of the general structure of the gravitational potentials when the field point $\boldsymbol{x}$ is in the wave zone, where $r := |\boldsymbol{x}| > \mathcal{R}$. Consulting Box 6.7 once more, we see that we can express $h^{\alpha\beta}_{\mathcal{N}}$ as the multipole expansion

$$h^{\alpha\beta}_{\mathcal{N}}(t, \boldsymbol{x}) = \frac{4G}{c^4}\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}\partial_L\left[\frac{1}{r}\int_{\mathscr{M}}\tau^{\alpha\beta}(\tau, \boldsymbol{x}')x'^L\, d^3x'\right], \tag{7.25}$$

in which $\tau := t - r/c$ is retarded time.

We first consider $h^{00}_{\mathcal{N}}$, and observe that the integral in Eq. (7.25) is just $c^2\mathcal{I}^L(\tau)$ as defined by Eqs. (7.16); the multipole moments are now evaluated at retarded time τ instead of time t. The first term ($\ell = 0$) in the series involves the monopole moment

$$M_0 := \mathcal{I}(\tau) = \int_{\mathscr{M}} c^{-2}\tau^{00}(\tau, \boldsymbol{x})\, d^3x, \tag{7.26}$$

and this represents the total mass contained within the near zone. Because of the conservation equations, we know that its time derivative can be converted to a surface integral on $\partial\mathscr{M}$, which can be shown to be small; the near-zone mass M_0 is therefore constant to a large degree of accuracy. The second term in the series involves

$$M_0 R_0^j := \mathcal{I}^j(\tau) = \int_{\mathscr{M}} c^{-2}\tau^{00}(\tau, \boldsymbol{x})x^j\, d^3x, \tag{7.27}$$

where R_0^j is the center-of-mass position associated with the domain $\mathscr{M}$. Its rate of change is related to the near-zone momentum

$$P_0^j := \int_{\mathscr{M}} c^{-1}\tau^{0j}(\tau, \boldsymbol{x})\, d^3x \tag{7.28}$$

by the conservation statement

$$\frac{d}{d\tau}\big(M_0 R_0^j\big) = P_0^j + \text{surface integral}\,, \tag{7.29}$$

and the momentum itself can be shown to satisfy

$$\frac{dP_0^j}{d\tau} = 0 + \text{surface integral}\,. \tag{7.30}$$

Because in each case the surface integral can be shown to be small, the total momentum is conserved to a large degree of accuracy, and the center-of-mass moves according to $d(M_0 R_0^j)/d\tau = P_0^j$. We may set $P_0^j = 0$ by working in the rest frame of the system, and set $R_0^j = 0$ by placing the center-of-mass at the spatial origin of the harmonic coordinates; the conservation equations ensure that R_0^j remains zero up to very small effects associated with the radiation of linear momentum. Thus, $h^{00}_{\mathscr{N}}$ consists of a static monopole piece plus time-dependent terms involving the quadrupole moment $\mathcal{I}^{jk}(\tau)$ and higher multipole moments.

Turning to $h^{0j}_{\mathscr{N}}$, and making use of the conservation identities of Eqs. (7.14a) and (7.14c), we can show that the $\ell = 0$ contribution to $h^{0j}_{\mathscr{N}}$ is of the form $(4G/c^3)r^{-1}\dot{\mathcal{I}}^j$ modulo surface terms; but since $\dot{\mathcal{I}}^j = P_0^j$ + surface integral, we find that this vanishes by virtue of our choice of reference frame. The $\ell = 1$ contribution involves $\int_{\mathscr{M}} \tau^{0j} x^k\, d^3x$, which according to Eq. (7.14c) can be converted to $\frac{1}{2}(\dot{\mathcal{I}}^{jk} - \epsilon^{mjk} J_0^m)$, where

$$J_0^m := \epsilon_{mjk} \int_{\mathscr{M}} x^j c^{-1} \tau^{0k}(\tau, \boldsymbol{x})\, d^3x \tag{7.31}$$

is the angular momentum contained within the near zone. The conservation identities can again be used to show that $dJ_0^m/d\tau$ vanishes up to a surface integral, so that $\boldsymbol{J}_0$ is constant except for a small radiative loss of angular momentum. Finally, looking at the $\ell = 0$ term in $h^{jk}_{\mathscr{N}}$ and using the identity of Eq. (7.14b), we find that we may convert it to $(2G/c^4)r^{-1}\ddot{\mathcal{I}}^{jk}$ modulo surface terms.

With these simplifications we obtain our final expression for $h^{\alpha\beta}_{\mathscr{N}}$ in the wave zone:

$$h^{00}_{\mathscr{N}} = \frac{4GM_0}{c^2 r} + \frac{4G}{c^2} \sum_{\ell=2}^{\infty} \frac{(-1)^\ell}{\ell!} \partial_L \left[\frac{\mathcal{I}^L(\tau)}{r}\right], \tag{7.32a}$$

$$\begin{aligned} h^{0j}_{\mathscr{N}} &= -\frac{2G}{c^3}\frac{(\boldsymbol{n}\times\boldsymbol{J}_0)^j}{r^2} - \frac{2G}{c^3}\partial_k \left[\frac{\dot{\mathcal{I}}^{jk}(\tau)}{r}\right] \\ &\quad + \frac{4G}{c^4}\sum_{\ell=2}^{\infty} \frac{(-1)^\ell}{\ell!} \partial_L \left[\frac{1}{r}\int_{\mathscr{M}} \tau^{0j}(\tau, \boldsymbol{x}')x'^L\, d^3x'\right], \end{aligned} \tag{7.32b}$$

$$h^{jk}_{\mathscr{N}} = \frac{2G}{c^4}\frac{\ddot{\mathcal{I}}^{jk}(\tau)}{r} + \frac{4G}{c^4}\sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{\ell!} \partial_L \left[\frac{1}{r}\int_{\mathscr{M}} \tau^{jk}(\tau, \boldsymbol{x}')x'^L\, d^3x'\right], \tag{7.32c}$$

in which overdots indicate differentiation with respect to $\tau = t - r/c$.

Still according to Box 6.7, we see that the wave-zone contribution $h^{\alpha\beta}_{\mathscr{W}}$ to the gravitational potentials is given by

$$h^{\alpha\beta}_{\mathscr{W}}(t,\boldsymbol{x}) = \frac{4G}{c^4}\frac{n^{\langle L\rangle}}{r}\left\{\int_0^{\mathcal{R}} ds f^{\alpha\beta}(\tau - 2s/c)A(s,r) + \int_{\mathcal{R}}^{\infty} ds f^{\alpha\beta}(\tau - 2s/c)B(s,r)\right\}, \tag{7.33}$$

when $\tau^{\alpha\beta}$ can be put in the specific form displayed in Eq. (7.19); the functions $A(s,r)$ and $B(s,r)$ are defined by Eq. (6.104). We shall learn how to evaluate these contributions below in Sec. 7.4, and then again in Chapter 11.

Box 7.2 **Multipole structure of the wave-zone metric**

By using extensions of the conservation identities (7.14), the wave-zone forms of the potentials $h^{\alpha\beta}_{\mathscr{N}}$ can be expressed elegantly in terms of a sequence of multipole moments. The general expressions are

$$h^{\alpha\beta}_{\mathscr{N}} = \frac{4G}{c^4}\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}\partial_L\left[\frac{1}{r}\mathcal{M}^{\alpha\beta L}(\tau)\right],$$

where

$$\begin{aligned}
\mathcal{M}^{00L} &= c^2\mathcal{I}^L,\\
\mathcal{M}^{0jL} &= \frac{c}{2(\ell+1)}\left(\dot{\mathcal{I}}^{jL} - \ell\epsilon^{mja_1}\mathcal{J}^{ma_2\cdots a_\ell}\right)\ (\mathrm{sym}\, a : L)\\
&\quad + \frac{1}{(\ell+1)}\oint_{\partial\mathscr{M}}\tau^{0m}x^{jL}\,dS_m,\\
\mathcal{M}^{jkL} &= \frac{1}{(\ell+1)(\ell+2)}\ddot{\mathcal{I}}^{jkL} + \frac{2}{(\ell+2)}\epsilon^{ma_1(j}\dot{\mathcal{J}}^{m|k)a_2\cdots a_\ell}\ (\mathrm{sym}\, a : L)\\
&\quad + \frac{8(\ell-1)}{(\ell+1)}\mathcal{P}^{jk(a_1a_2\cdots a_\ell)}\\
&\quad + \frac{1}{(\ell+1)(\ell+2)}\oint_{\partial\mathscr{M}}\left[\tau^{mn}\partial_n(x^{jkL}) + \partial_\tau\tau^{0m}x^{jkL}\right]dS_m\\
&\quad - \frac{2}{(\ell+2)}\oint_{\partial\mathscr{M}}\left[\tau^{n[a_1}x^{j]ka_2\cdots a_\ell} + (j \rightleftharpoons k)\right]dS_n\ (\mathrm{sym}\, a : L),
\end{aligned}$$

where $\mathcal{I}^L$ and $\mathcal{J}^{jL}$ are defined in Eqs. (7.16), and

$$\mathcal{P}^{jkabL} := \int_{\mathscr{M}} x^{[a}\tau^{j][k}x^{b]L}\,d^3x.$$

The notation $(\mathrm{sym}\, a : L)$ means symmetrize on all ℓ a-indices.

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7.1.5 Toward two iterations of the field equations

As we pointed out back in Sec. 6.2.3, to achieve the second post-Minkowskian approximation to the gravitational potentials $h^{\alpha\beta}$, we must carry out two iterations of the relaxed field equations and then impose the gauge condition/conservation statement. In other words, we must solve the wave equation $\Box h^{\alpha\beta} = -(16\pi G/c^4)\tau_1^{\alpha\beta}$ for the potentials $h_2^{\alpha\beta}$ and then impose the gauge condition $\partial_\beta h_2^{\alpha\beta} = 0$ or the conservation equation $\partial_\beta \tau_1^{\alpha\beta} = 0$. The starting point of these computations is construction of the effective energy-momentum pseudotensor $\tau_1^{\alpha\beta}$, which depends on the fluid's energy-momentum tensor $T^{\alpha\beta}$ and the potentials generated during the *first iteration* of the relaxed field equations. Our very first task, therefore, is to perform the first iteration and obtain $\tau_1^{\alpha\beta}$.

7.2 First iteration

In this section we complete the first iteration of the relaxed field equations to obtain the gravitational potentials $h_1^{\alpha\beta}$. Our goal is to perform the computation to a degree of accuracy that is sufficient for the preparation of the second iteration, to be carried out in Secs. 7.3 and 7.4.

7.2.1 Energy-momentum tensor

In the first iteration of the field equations we replace $g_{\alpha\beta}$ by $\eta_{\alpha\beta}$ in the energy-momentum tensor of Eq. (7.1), and in the normalization condition for the velocity four-vector. Similarly, we set $\sqrt{-g} = 1$ in Eq. (7.4). We find that $\gamma = (1 - v^2/c^2)^{-1/2} = 1 + \frac{1}{2}(v/c)^2 + O(c^{-4})$, and Eq. (7.4) becomes

$$\rho = \left[1 - \frac{1}{2}(v/c)^2 + O(c^{-4})\right]\rho^*. \tag{7.34}$$

The components of the energy-momentum tensor are then

$$c^{-2}T_0^{00} = \rho^*\left[1 + \frac{1}{c^2}\left(\frac{1}{2}v^2 + \Pi\right) + O(c^{-4})\right], \tag{7.35a}$$

$$c^{-1}T_0^{0j} = \rho^* v^j\left[1 + \frac{1}{c^2}\left(\frac{1}{2}v^2 + \Pi + p/\rho^*\right) + O(c^{-4})\right], \tag{7.35b}$$

$$T_0^{jk} = \rho^* v^j v^k + p\,\delta^{jk} + O(c^{-2}). \tag{7.35c}$$

They are written as post-Newtonian expansions in flat spacetime, and these include both Newtonian and post-Newtonian contributions; terms occurring at 2PN order are neglected. Because they do not yet include 1PN terms involving the gravitational potentials, which will appear during the second iteration of the field equations, these post-Newtonian expansions are incomplete.

7.2.2 Near zone

We first take the field point $\boldsymbol{x}$ to be in the near zone, so that $r < \mathcal{R}$. To achieve the first iteration of the relaxed field equations, we set $\tau^{\alpha\beta} = T_0^{\alpha\beta}$ and make the substitution within Eqs. (7.13). Because the energy-momentum tensor is confined to the near zone, there is no need to truncate the integrals to the near-zone domain $\mathscr{M}$; they are naturally truncated to the volume occupied by the matter distribution. And because $T_0^{\alpha\beta}$ does not extend to the wave zone, the potentials $h^{\alpha\beta}_{\mathscr{W}}$ vanish, and $h^{\alpha\beta} = h^{\alpha\beta}_{\mathscr{N}}$.

As we shall see below in Sec. 7.3, for the purposes of preparing the second iteration of the field equations it is sufficient to compute h_1^{00} to order c^{-2}, h_1^{0j} to order c^{-3}, and to neglect h_1^{jk} because it is of order c^{-4}. This requirement implies that we can truncate Eqs. (7.35) to

$$c^{-2}T_0^{00} = \rho^* + O(c^{-2}), \tag{7.36a}$$

$$c^{-1}T_0^{0j} = \rho^* v^j + O(c^{-2}), \tag{7.36b}$$

$$T^{jk} = O(1). \tag{7.36c}$$

Making the substitution within Eq. (7.13) reveals that the potentials are given by

$$h_1^{00} = \frac{4}{c^2}U + O(c^{-4}), \tag{7.37a}$$

$$h_1^{0j} = \frac{4}{c^3}U^j + O(c^{-4}), \tag{7.37b}$$

$$h_1^{jk} = O(c^{-4}), \tag{7.37c}$$

in which U is a Newtonian potential defined by

$$U(t, \boldsymbol{x}) = G\int \frac{\rho^*(t, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}\, d^3x', \qquad \nabla^2 U = -4\pi G\rho^*, \tag{7.38}$$

in terms of the rescaled mass density ρ^*, and U^j is a vector potential defined by

$$U^j(t, \boldsymbol{x}) = G\int \frac{\rho^* v^j(t, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}\, d^3x', \qquad \nabla^2 U^j = -4\pi G\rho^* v^j, \tag{7.39}$$

in terms of the mass-current density $\rho^* v^j$. It is useful to note that by virtue of the continuity equation (7.3), the potentials satisfy

$$\partial_t U + \partial_j U^j = 0 \tag{7.40}$$

as a matter of identity.

Note that in Eq. (7.37), the corrections to h_1^{00} first occur at order c^{-4}. The expansion of Eq. (7.13), however, contains a term at order c^{-3} proportional to

$$\frac{d}{dt}\int \rho^*\, d^3x.$$

This vanishes because $m := \int \rho^*\, d^3x$, the total rest-mass within the fluid is conserved by virtue of Eq. (7.3). As was pointed out back in Sec. 7.1.1, the conservation of rest-mass is a basic kinematical requirement, quite divorced from any dynamical requirement based on energy-momentum conservation. This is an important point, because we recall that we are not at liberty to impose the conservation equations $\partial_\beta \tau^{\alpha\beta} = 0$ during the first iteration of the relaxed field equations; for this we must await the second iteration. With this in mind, you will notice that the corrections to h_1^{0j} first occur at order c^{-4}; this represents a term proportional to

$$\frac{d}{dt}\int \rho^* v^j \, d^3x$$

in the expansion of Eq. (7.13). The integral is the total momentum at Newtonian order, and it is tempting to declare that this term should vanish by virtue of momentum conservation. This temptation, however, must be resisted during the first iteration.

The gravitational potentials may be inserted within Eqs. (7.24) to obtain the near-zone metric. We obtain

$$g_{00}^1 = -1 + \frac{2}{c^2}U + O(c^{-4}), \tag{7.41a}$$

$$g_{0j}^1 = -\frac{4}{c^3}U^j + O(c^{-4}), \tag{7.41b}$$

$$g_{jk}^1 = \Big(1 + \frac{2}{c^2}U\Big)\delta_{jk} + O(c^{-4}), \tag{7.41c}$$

and the metric determinant is $(-g_1) = 1 + 4U/c^2 + O(c^{-4})$. Recalling our discussion in Sec. 7.1.3, we see that this metric is not sufficiently accurate to obtain the motion of a test particle at post-Newtonian order, because it lacks the $O(c^{-4})$ contributions to g_{00}. It is sufficiently accurate, however, to serve as input in the second iteration of the relaxed field equations.

7.2.3 Wave zone

We next take the field point $\boldsymbol{x}$ to be in the wave zone, so that $r > \mathcal{R}$. To achieve the first iteration of the relaxed field equations, we could in principle set $\tau^{\alpha\beta} = T_0^{\alpha\beta}$, make the substitution within Eqs. (7.25), and evaluate the multipole moments explicitly. There is, however, no immediate need to proceed in this way. We can instead keep things simple by making direct use of Eqs. (7.25) and keeping the multipole moments unevaluated until we have completed the second iteration. An aspect of $\tau^{\alpha\beta}$ that we can incorporate is that it does not extend beyond the near zone; this implies that $h_{\mathscr{W}}^{\alpha\beta}$ vanishes, so that $h^{\alpha\beta} = h_{\mathscr{N}}^{\alpha\beta}$.

As we shall see below in Sec. 7.4, only h_1^{00} is required in the preparation of the second iteration. It is given by

$$h_1^{00} = \frac{4G}{c^2}\left\{\frac{\mathcal{I}(\tau)}{r} - \partial_j\left[\frac{\mathcal{I}^j(\tau)}{r}\right] + \frac{1}{2}\partial_{jk}\left[\frac{\mathcal{I}^{jk}(\tau)}{r}\right] + \cdots\right\}, \tag{7.42}$$

in which $\mathcal{I}^L(\tau) := \int_{\mathscr{M}} c^{-2}\tau^{00}(\tau, \boldsymbol{x})x^L \, d^3x$ are the multipole moments of the density $c^{-2}\tau^{00}$, expressed as functions of retarded time $\tau = t - r/c$. Note that we keep the dipole-moment term in the expansion, in spite of the fact that $\mathcal{I}^j$ will eventually be set equal to zero by a coordinate choice, as we indicated back in Sec. 7.1.4. The reason is that the ability to set $\mathcal{I}^j = 0$ relies on the conservation equation $\partial_\beta \tau^{\alpha\beta} = 0$, which we are not at liberty to impose during the first iteration.

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Counting post-Newtonian orders is more subtle in the wave zone than it is in the near zone. The monopole term on the right-hand side of Eq. (7.42) is evidently of order $Gm_c/(c^2r)$, and we naturally assign a 0PN order to this term. To see about the dipole term, we perform the differentiation and express it as

$$-\frac{4G}{c^2}\partial_j\left[\frac{\mathcal{I}^j(\tau)}{r}\right] = \frac{4G}{c^2}\left(\frac{\dot{\mathcal{I}}^j}{cr} + \frac{\mathcal{I}^j}{r^2}\right)n_j\,, \tag{7.43}$$

in which $n^j := x^j/r$. Noting that $\mathcal{I}^j$ scales as $m_c r_c$, this term is of order

$$\frac{G}{c^2}m_c r_c\left(\frac{1}{ct_c r} + \frac{1}{r^2}\right) = \frac{Gm_c}{c^2 r}\frac{r_c}{ct_c}\left(1 + \frac{ct_c}{r}\right). \tag{7.44}$$

This is smaller than $Gm_c/(c^2r)$ by a factor of order $(v_c/c)(1 + \lambda_c/r)$. The second factor is of order unity in the wave zone, and we conclude that the dipole term is smaller than the monopole term by a factor of order v_c/c. To this term we therefore assign a 0.5PN order. We do this in spite of the fact that the second term on the right of Eq. (7.43) is formally of Newtonian order. In the near zone, but outside the distribution of matter, this term would give the standard dipole contribution to the Newtonian potential, which normally would be set equal to zero by a suitable choice of coordinates. But because it falls off as r^{-2} and we are looking in the wave zone at distances $r > \lambda_c = r_c(c/v_c)$, it has decreased in size to such an extent that it is now comparable to (or even smaller than) the 0.5PN term produced by the time derivative of $\mathcal{I}^j$.

A simple extension of this argument reveals that the quadrupole term in h_1^{00} must be assigned a 1PN order. The octupole term, which would occur next in Eq. (7.42), gives a contribution at 1.5PN order, and the post-Newtonian counting becomes clear: an ℓ-pole term contributes at $(\ell/2)$PN order to the gravitational potential.

7.3 Second iteration: Near zone

In this section we face the challenging task of completing the second iteration of the relaxed field equations. Here we take the field point $\boldsymbol{x}$ to be in the near zone, so that $r < \mathcal{R}$. The wave zone will be considered next, in Sec. 7.4.

7.3.1 Effective energy momentum pseudotensor

Our first order of business is to use the potentials obtained in the first iteration to construct the effective energy-momentum pseudotensor of Eq. (6.52),

$$\tau^{\alpha\beta} = (-g)\big(T^{\alpha\beta} + t_{\rm LL}^{\alpha\beta} + t_{\rm H}^{\alpha\beta}\big), \tag{7.45}$$

with the Landau–Lifshitz contribution defined by Eq. (6.5), and the harmonic contribution defined by Eq. (6.53).

We begin by updating our expression for $T^{\alpha\beta}$, the fluid's energy-momentum tensor, which was given an incomplete post-Newtonian expansion back in Eq. (7.35). We return to Eq. (7.1) and substitute the near-zone metric displayed in Eq. (7.41). We also involve

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CUUK2552-07 CUUK2552-Poisson 978 1 107 03286 6 December 16, 2013 10:51

this metric in the normalization condition for u^α, and update our expression for γ to $1 + \frac{1}{2}(v/c)^2 + U/c^2 + O(c^{-4})$, which now incorporates the Newtonian potential U. Equation (7.34) becomes

$$\rho = \left[1 - \frac{1}{c^2}\left(\frac{1}{2}v^2 + 3U\right) + O(c^{-4})\right]\rho^*, \tag{7.46}$$

and the components of the energy-momentum tensor are now

$$c^{-2}(-g)T_1^{00} = \rho^*\left[1 + \frac{1}{c^2}\left(\frac{1}{2}v^2 + 3U + \Pi\right) + O(c^{-4})\right], \tag{7.47a}$$

$$c^{-1}(-g)T_1^{0j} = \rho^* v^j\left[1 + \frac{1}{c^2}\left(\frac{1}{2}v^2 + 3U + \Pi + p/\rho^*\right) + O(c^{-4})\right], \tag{7.47b}$$

$$(-g)T_1^{jk} = \rho^* v^j v^k + p\,\delta^{jk} + O(c^{-2}). \tag{7.47c}$$

We have multiplied $T_1^{\alpha\beta}$ by $(-g)$ because this is the combination that appears in $\tau^{\alpha\beta}$.

The hardest piece of the calculation by far (and this is always true) is the computation of $(-g)t_{\rm LL}^{\alpha\beta}$ to the appropriate degree of accuracy. To match the accuracy achieved in Eqs. (7.47) we need $c^{-2}(-g)t_{\rm LL}^{00}$ to orders $O(1)$ and $O(c^{-2})$, $c^{-1}(-g)t_{\rm LL}^{0j}$ to order $O(1)$ and $O(c^{-2})$, and $(-g)t_{\rm LL}^{jk}$ to order $O(1)$. To pluck out of Eq. (6.5) the terms of relevant orders, we use the facts recorded in Eq. (7.9), that the potentials scale as $h^{00} = O(c^{-2})$, $h^{0j} = O(c^{-3})$, and $h^{jk} = O(c^{-4})$. In addition, we use the property that $\partial_0 h^{00}$ is of order c^{-1} relative to $\partial_j h^{00}$. The dominant piece of $(-g)t_{\rm LL}^{\alpha\beta}$ will therefore come from $\partial_j h^{00} = 4\partial_j U/c^2$.

Armed with these observations, the reduction of $(-g)t_{\rm LL}^{\alpha\beta}$ to something manageable is well within reach. Let us, for example, examine the term

$$\frac{1}{4}\left(2g^{\alpha\lambda}g^{\beta\mu} - g^{\alpha\beta}g^{\lambda\mu}\right)g_{\nu\rho}g_{\sigma\tau}\partial_\lambda h^{\nu\tau}\partial_\mu h^{\rho\sigma}$$

on the right-hand side of Eq. (6.5), in which we have replaced $\mathfrak{g}^{\alpha\beta}$ by $\eta^{\alpha\beta} - h^{\alpha\beta}$. A first source of simplification arises from the fact that each occurrence of $g_{\alpha\beta}$ can be replaced by $\eta_{\alpha\beta}$; this comes about because each factor of $h^{\alpha\beta}$ contributes a power of G, and we need to compute $(-g)t_{\rm LL}^{\alpha\beta}$ to order G^2 in the second post-Minkowskian approximation. A second source of simplification comes from the fact that at leading order, we can retain terms that involve $\partial_j h^{00}$ only. At this stage the previous expression becomes

$$\frac{1}{4}\left(2\eta^{\alpha j}\eta^{\beta k} - \eta^{\alpha\beta}\delta^{jk}\right)\partial_j h^{00}\partial_k h^{00},$$

and it gives rise to a contribution $\frac{1}{4}\partial_j h^{00}\partial^j h^{00}$ to $(-g)t_{\rm LL}^{00}$, and a contribution $\frac{1}{2}\partial^j h^{00}\partial^k h^{00} - \frac{1}{4}\delta^{jk}\partial_n h^{00}\partial^n h^{00}$ to $(-g)t_{\rm LL}^{jk}$; there is no contribution to $(-g)t_{\rm LL}^{0j}$.

Keeping track of all terms that make up $(-g)t_{\rm LL}^{\alpha\beta}$, we eventually arrive at the expressions

$$\frac{16\pi G}{c^4}(-g)t_{\rm LL}^{00} = -\frac{7}{8}\partial_j h^{00}\partial^j h^{00} + O(c^{-6}), \tag{7.48a}$$

$$\frac{16\pi G}{c^4}(-g)t_{\rm LL}^{0j} = \frac{3}{4}\partial^j h^{00}\partial_0 h^{00} + \left(\partial^j h^{0k} - \partial^k h^{0j}\right)\partial_k h^{00} + O(c^{-7}), \tag{7.48b}$$

$$\frac{16\pi G}{c^4}(-g)t_{\rm LL}^{jk} = \frac{1}{4}\partial^j h^{00}\partial^k h^{00} - \frac{1}{8}\delta^{jk}\partial_n h^{00}\partial^n h^{00} + O(c^{-6}). \tag{7.48c}$$

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CUUK2552-07 CUUK2552-Poisson 978 1 107 03286 6 December 16, 2013 10:51

These results are sufficiently accurate for our immediate purposes. At a later stage, however, we shall need additional accuracy in our expression for $(-g)t^{jk}_{\rm LL}$, and we record this improved expression here:

$$\begin{aligned}\frac{16\pi G}{c^4}(-g)t^{jk}_{\rm LL} &= \frac{1}{4}\big(1-2h^{00}\big)\partial^j h^{00}\partial^k h^{00} - \frac{1}{8}\delta^{jk}\big(1-2h^{00}\big)\partial_n h^{00}\partial^n h^{00} \\ &\quad - \partial^j h^{0n}\partial^k h^0_{\ n} + \partial^j h^{0n}\partial_n h^{0k} + \partial^k h^{0n}\partial_n h^{0j} - \partial_n h^{0j}\partial^n h^{0k} \\ &\quad + \partial^j h^{00}\partial_0 h^{0k} + \partial^k h^{00}\partial_0 h^{0j} + \frac{1}{4}\partial^j h^{00}\partial^k h^n_{\ n} + \frac{1}{4}\partial^k h^{00}\partial^j h^n_{\ n} \\ &\quad + \delta^{jk}\bigg[-\frac{3}{8}\big(\partial_0 h^{00}\big)^2 - \partial_n h^{00}\partial_0 h^{0n} - \frac{1}{4}\partial_n h^{00}\partial^n h^p_{\ p} \\ &\qquad + \frac{1}{2}\partial_n h^0_{\ p}\big(\partial^n h^{0p} - \partial^p h^{0n}\big)\bigg] + O(c^{-8}). \end{aligned} \tag{7.49}$$

It should be noted that this incorporates corrections of order c^{-2} relative to the leading-order expression of Eq. (7.48), and that to be consistent, we have terms such as $h^{00}\partial^j h^{00}\partial^k h^{00}$ that contain an additional power of the gravitational constant G.

With the substitutions of Eqs. (7.37), the Landau–Lifshitz pseudotensor becomes

$$c^{-2}(-g)t^{00}_{\rm LL} = -\frac{1}{4\pi Gc^2}\left(\frac{7}{2}\partial_j U\partial^j U\right) + O(c^{-4}), \tag{7.50a}$$

$$c^{-1}(-g)t^{0j}_{\rm LL} = \frac{1}{4\pi Gc^2}\Big[3\partial_t U\partial^j U + 4\big(\partial^j U^k - \partial^k U^j\big)\partial_k U\Big] + O(c^{-4}), \tag{7.50b}$$

$$(-g)t^{jk}_{\rm LL} = \frac{1}{4\pi G}\left(\partial^j U\partial^k U - \frac{1}{2}\delta^{jk}\partial_n U\partial^n U\right) + O(c^{-2}). \tag{7.50c}$$

To better understand the importance of these contributions to $\tau^{\alpha\beta}$, we estimate the order of magnitude of $c^{-2}(-g)t^{00}_{\rm LL}$ relative to ρ^*, the dominant contribution to $c^{-2}\tau^{00}$. We re-introduce the scaling quantities m_c, r_c, and v_c, and estimate the pseudotensor within the matter distribution. We have that $\rho^* \sim m_c/r_c^3$ and $U \sim Gm_c/r_c$. After differentiation we get $\partial_j U \sim Gm_c/r_c^2$, and all this produces

$$\frac{(-g)t^{00}_{\rm LL}}{\rho^* c^2} \sim \frac{Gm_c}{c^2 r_c}. \tag{7.51}$$

Since motion within the fluid is governed by gravity, we can rely on the virial theorem and claim that $Gm_c/r_c \sim v_c^2$. The end result is that $c^{-2}(-g)t^{00}_{\rm LL}$ is a quantity of order $(v_c/c)^2$ relative to ρ^*; it is comparable to the other 1PN terms that are displayed in Eq. (7.47).

The easiest piece of the calculation by far (and this is always true) is the computation of $(-g)t^{\alpha\beta}_{\rm H}$ to the required degree of accuracy. Using the information gathered previously, Eq. (6.53) returns

$$\frac{16\pi G}{c^4}(-g)t^{00}_{\rm H} = O(c^{-6}), \tag{7.52a}$$

$$\frac{16\pi G}{c^4}(-g)t^{0j}_{\rm H} = O(c^{-7}), \tag{7.52b}$$

$$\frac{16\pi G}{c^4}(-g)t^{jk}_{\rm H} = O(c^{-6}). \tag{7.52c}$$

These expressions should be compared with Eqs. (7.48); they imply that the harmonic pseudotensor makes no relevant contribution to $\tau^{\alpha\beta}$. For later reference we record the improved expression

$$\frac{16\pi G}{c^4}(-g)t_{\rm H}^{jk} = -h^{00}\partial_{00}h^{jk} + O(c^{-8}) \tag{7.53}$$

for the spatial components of the pseudotensor.

Collecting results, we have obtained

$$c^{-2}\tau_1^{00} = \rho^*\biggl[1 + \frac{1}{c^2}\biggl(\frac{1}{2}v^2 + 3U + \Pi\biggr)\biggr] - \frac{1}{4\pi Gc^2}\biggl(\frac{7}{2}\partial_j U\partial^j U\biggr) + O(c^{-4}), \tag{7.54a}$$

$$\begin{aligned} c^{-1}\tau_1^{0j} &= \rho^* v^j\biggl[1 + \frac{1}{c^2}\biggl(\frac{1}{2}v^2 + 3U + \Pi + p/\rho^*\biggr)\biggr] \\ &\quad + \frac{1}{4\pi Gc^2}\Bigl[3\partial_t U\partial^j U + 4\bigl(\partial^j U^k - \partial^k U^j\bigr)\partial_k U\Bigr] + O(c^{-4}), \end{aligned} \tag{7.54b}$$

$$\tau_1^{jk} = \rho^* v^j v^k + p\,\delta^{jk} + \frac{1}{4\pi G}\biggl(\partial^j U\partial^k U - \frac{1}{2}\delta^{jk}\partial_n U\partial^n U\biggr) + O(c^{-2}), \tag{7.54c}$$

for the effective energy-momentum pseudotensor.

7.3.2 Energy-momentum conservation

At this stage of our development of the second post-Minkowskian approximation, we may impose the conservation equations

$$c^{-2}\partial_t\tau_1^{00} + c^{-1}\partial_j\tau_1^{0j} = 0, \qquad c^{-1}\partial_t\tau_1^{0j} + \partial_k\tau_1^{jk} = 0, \tag{7.55}$$

before calculating the second-iterated potentials $h_2^{\alpha\beta}$. At leading order the energy equation reproduces Eq. (7.3); not surprisingly, a statement of rest-mass conservation is included in the statement of energy conservation. The equation brings additional information at order c^{-2}, a statement of energy conservation for all relevant forms of fluid energy: kinetic, internal, and gravitational. We shall return to this theme below.

The momentum equation is equally informative. Using Eqs. (7.54), we have

$$\begin{aligned} c^{-1}\partial_t\tau_1^{0j} &= (\partial_t\rho^*)v^j + \rho^*\partial_t v^j + O(c^{-2}) \\ &= -v^j\partial_k(\rho^* v^k) + \rho^*\frac{dv^j}{dt} - \rho^* v^k\partial_k v^j + O(c^{-2}), \end{aligned} \tag{7.56}$$

where we have involved Eq. (7.3) and the definition of the total time derivative: $dv^j/dt = \partial_t v^j + v^k\partial_k v^j$. We also have

$$\partial_k\tau_1^{jk} = v^j\partial_k(\rho^* v^k) + \rho^* v^k\partial_k v^j + \partial^j p + \frac{1}{4\pi G}(\partial^j U)\nabla^2 U + O(c^{-2}). \tag{7.57}$$

Making the substitutions into Eq. (7.55), and replacing $\nabla^2 U$ by $-4\pi G\rho^*$, we arrive at

$$\rho^*\frac{dv^j}{dt} = \rho^*\partial^j U - \partial^j p + O(c^{-2}). \tag{7.58}$$

This is Euler's equation, which governs the dynamics of our perfect fluid at leading order in a post-Newtonian expansion. It was first obtained on the basis of Newtonian theory in

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CUUK2552-07 CUUK2552-Poisson 978 1 107 03286 6 December 16, 2013 10:51

Chapter 1, and indeed, the foregoing computations have already been presented (in reverse order) in Sec. 1.4.4.

Recalling our discussion of the iteration procedure in Sec. 6.2.3, we observe that Euler's equation (i.e. Newtonian gravity) is the consequence of $\partial_\beta \tau_1^{\alpha\beta} = 0$, the conservation equation that goes along with the *second iteration* of the relaxed field equations. Performing a single iteration is not sufficient to produce this dynamics, because the equations of motion that are compatible with the first iteration, derived from the conservation equation $\partial_\beta \tau_0^{\alpha\beta} = 0$, do *not contain gravitational interactions*. This observation was also made in the context of the linearized approximation to general relativity, back in Sec. 5.5. So formally, a second iteration of the relaxed field equations is required to obtain the Newtonian equations of motion. Similarly, a third iteration is required to find the post-Newtonian equations of motion, and so on. But as we also discussed back in Sec. 6.2.3, the conservation equation compatible with the nth iteration requires ingredients that are collected during the $(n-1)$th iteration, and it can be formulated *before* completing the nth iteration to obtain the gravitational potentials. In practice, therefore, we may obtain the Newtonian equations of motion on the basis of the first-iterated potentials; the post-Newtonian equations on the basis of the second-iterated potentials, and so on.

As we saw back in Sec. 6.1.4, the local conservation equations (7.55) imply the existence of globally conserved quantities. From Eq. (6.36) we have the *total mass*

$$M := \frac{1}{c^2} \int (-g)\big(T^{00} + t_{\rm LL}^{00}\big)\, d^3x, \tag{7.59}$$

and from Eq. (6.37) we have the *total momentum*

$$P^j := \frac{1}{c} \int (-g)\big(T^{0j} + t_{\rm LL}^{0j}\big)\, d^3x. \tag{7.60}$$

In addition, it is useful to re-introduce the vector

$$R^j := \frac{1}{Mc^2} \int (-g)\big(T^{00} + t_{\rm LL}^{00}\big) x^j\, d^3x, \tag{7.61}$$

which denotes the position of the center-of-mass; this was first defined by Eq. (6.39). We recall that R^j is related to the total momentum by the equation $M dR^j/dt = P^j$, and that by adopting the center-of-mass frame of the spacetime, we can set both P^j and R^j to zero. It is worth pointing out that since $(-g)t_{\rm H}^{\alpha\beta}$ makes no relevant contribution to $\tau_1^{\alpha\beta}$ at this order, as we saw back in Eq. (7.52), the conserved quantities associated with $(-g)(T^{\alpha\beta} + t_{\rm LL}^{\alpha\beta})$ are the same as those associated with $\tau^{\alpha\beta}$.

The global quantities M, P^j, and R^j are defined in terms of integrals that extend over all space. We may still, however, evaluate them with the near-zone information available to us now, because their expressions turn out to be insensitive to the wave-zone aspects of the integrals. To evaluate M we insert our previous expression for $c^{-2}\tau_1^{00}$ within Eq. (7.59), which we truncate to the near-zone domain $\mathcal{M}$. The term involving U is handled as follows. We write

$$\partial_j U \partial^j U = \partial_j\big(U\partial^j U\big) - U\nabla^2 U = \partial_j\big(U\partial^j U\big) + 4\pi G\rho^* U \tag{7.62}$$

and observe that the first term gives rise to a surface integral that must be evaluated at $r = \mathcal{R}$; it makes an $\mathcal{R}$-dependent contribution to M that cancels out when the wave-zone portion of the integral is added to the near-zone portion. Collecting results, we arrive at

$$M = \int \rho^* \left[1 + \frac{1}{c^2} \left(\frac{1}{2} v^2 - \frac{1}{2} U + \Pi \right) \right] d^3x + O(c^{-4}) \tag{7.63}$$

for the total mass. The integral of ρ^* is m, the total rest-mass of the fluid, which is separately conserved. The integral of $\frac{1}{2}\rho^* v^2$ is $\mathcal{T}$, the fluid's total kinetic energy. The integral of $-\frac{1}{2}\rho^* U$ is Ω, the gravitational potential energy. And finally, the integral of $\rho^*\Pi = \epsilon + O(c^{-2})$ is E_{int}, the total internal energy stored within the fluid. The sum of $\mathcal{T}$, Ω, and E_{int} is the total energy E, and this was shown to be constant (by virtue of Euler's equation and the first law of thermodynamics) back in Sec. 1.4.3. The total mass can therefore be expressed as $M = m + E/c^2 + O(c^{-4})$, and this equation possesses a clear interpretation: The total mass of the spacetime is a measure of all forms of energy, including rest-mass, kinetic, gravitational, and internal energies.

Similar manipulations reveal that R^j can be expressed as

$$R^j = \frac{1}{M} \int \rho^* x^j \left[1 + \frac{1}{c^2} \left(\frac{1}{2} v^2 - \frac{1}{2} U + \Pi \right) \right] d^3x + O(c^{-4}), \tag{7.64}$$

and Eq. (7.60) becomes

$$\begin{aligned} P^j = &\int \rho^* v^j \left[1 + \frac{1}{c^2} \left(\frac{1}{2} v^2 - \frac{1}{2} U + \Pi + p/\rho^* \right) \right] d^3x \\ &- \frac{1}{2c^2} \int \rho^* x^j \left(\partial_t U - v^k \partial_k U \right) d^3x + O(c^{-4}). \end{aligned} \tag{7.65}$$

The leading-order piece of the total momentum was shown to be constant (by virtue of Euler's equation) back in Sec. 1.4.3 of Chapter 1; with this improved expression the momentum is conserved to order c^{-2}.

It is instructive to examine the relationship between the total mass M, which is known to correspond to the ADM mass of the spacetime, and the near-zone mass M_0, defined by Eq. (7.26),

$$M_0 = \int_{\mathcal{M}} c^{-2} \tau^{00} \, d^3x, \tag{7.66}$$

which appears in the expression of Eq. (7.32) for h^{00} in the wave zone. It is easy to see that

$$M_0 = M + O(c^{-4}). \tag{7.67}$$

This follows because the integrands for M and M_0 differ by $(-g)t^{00}_{\text{H}}$, which makes no contribution at 1PN order, and because the wave-zone portion of the integral defining M makes no $\mathcal{R}$-independent contribution to the mass. Examining the relationship at higher post-Newtonian orders, we find that subtle differences between M_0 and M appear at order c^{-5}; these are explored in Exercise 7.8.

Similar manipulations reveal that

$$R_0^j = R^j + O(c^{-4}) \tag{7.68}$$

and

$$P_0^j = P^j + O(c^{-4}), \tag{7.69}$$

in which R_0^j is the position of the near-zone center-of-mass introduced in Eq. (7.27), and P_0^j is the near-zone momentum introduced in Eq. (7.28). These equalities imply that at 1PN order, a coordinate choice that enforces $R^j = 0 = P^j$ also enforces $R_0^j = 0 = P_0^j$.

7.3.3 Near-zone contribution to potentials

Armed with Eq. (7.54) for $\tau_1^{\alpha\beta}$, we are now ready to solve the relaxed field equations for the second-iterated potentials $h^{\alpha\beta} = h_{\mathscr{N}}^{\alpha\beta} + h_{\mathscr{W}}^{\alpha\beta}$. In this section we focus on the near-zone contribution $h_{\mathscr{N}}^{\alpha\beta}$, insert $\tau_1^{\alpha\beta}$ within Eqs. (7.15), and express the results in a convenient form. The spatial components h^{jk} require special care, because as we have observed in Sec. 7.1.3, the spatial trace h^{kk} contributes to the spacetime metric at 1PN order, while the remaining components contribute only at 2PN order. With this in mind, it is helpful to decompose the potentials into a "scalar class" comprising h^{00} and h^{kk}, a "vector class" comprising h^{0j}, and a "tensor class" comprising h^{jk}.

Scalar class

We begin with the computation of h^{00} and h^{kk}. Examining Eqs. (7.54), we observe that both τ_1^{00} and τ_1^{kk} contain a contribution proportional to $\partial_j U \partial^j U$, which does not have compact support. It is useful to re-express these terms by exploiting the identity

$$\nabla^2 U^2 = 2\partial_j U \partial^j U + 2U\nabla^2 U, \tag{7.70}$$

in which we may insert Poisson's equation $\nabla^2 U = -4\pi G\rho^*$. In this way we obtain

$$c^{-2}\tau_1^{00} = \rho^*\left[1 + \frac{1}{c^2}\left(\frac{1}{2}v^2 - \frac{1}{2}U + \Pi\right)\right] - \frac{7}{16\pi G c^2}\nabla^2 U^2 + O(c^{-4}) \tag{7.71}$$

and

$$\tau_1^{kk} = \rho^*\left(v^2 - \frac{1}{2}U\right) + 3p - \frac{1}{16\pi G}\nabla^2 U^2 + O(c^{-2}) \tag{7.72}$$

for the relevant components of the energy-momentum pseudotensor.

Consulting Eq. (7.15), we see that the leading terms in both $h_{\mathscr{N}}^{00}$ and $h_{\mathscr{N}}^{kk}$ are Poisson integrals constructed from $c^{-2}\tau^{00}$ and τ^{kk}. To evaluate these we must distinguish between the pieces of the source functions that have compact support (those that are tied to the fluid variables), and those that depend on the Newtonian potential and extend beyond the matter distribution. To handle the compact-support pieces we introduce the potentials

$$\psi(t, \boldsymbol{x}) := G\int \frac{\rho^{*\prime}\left(\frac{3}{2}v'^2 - U' + \Pi'\right) + 3p'}{|\boldsymbol{x} - \boldsymbol{x}'|}\, d^3x', \tag{7.73a}$$

$$V(t, \boldsymbol{x}) := G\int \frac{\rho^{*\prime}\left(v'^2 - \frac{1}{2}U'\right) + 3p'}{|\boldsymbol{x} - \boldsymbol{x}'|}\, d^3x', \tag{7.73b}$$

in which primed quantities such as $\rho^{*\prime}$ indicate that the fluid variables are expressed as functions of t and $\boldsymbol{x}'$. These satisfy the Poisson equations

$$\nabla^2\psi = -4\pi G\rho^*\left(\frac{3}{2}v^2 - U + \Pi + 3p/\rho^*\right), \tag{7.74a}$$

$$\nabla^2 V = -4\pi G\rho^*\left(v^2 - \frac{1}{2}U + 3p/\rho^*\right). \tag{7.74b}$$

With this notation we see that at leading order, the compact-support piece of $h^{00}_{\mathcal{N}}$ is given by $4U/c^2 + 4(\psi - V)/c^4$, while the compact-support piece of $h^{kk}_{\mathcal{N}}$ is $4V/c^4$; this choice of notation is motivated by the fact that once the potentials are inserted within the near-zone metric of Eq. (7.24), the leading-order, compact-support piece of g_{00} will involve only U and ψ.

Turning next to the Poisson integral involving $\nabla^2 U^2$, we evaluate it by making repeated use of integration by parts:

$$\begin{aligned}
\frac{1}{4\pi}\int_{\mathcal{M}} \frac{\nabla'^2 U'^2}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x' &= \frac{1}{4\pi}\oint_{\partial\mathcal{M}} \frac{\partial'^j U'^2}{|\boldsymbol{x}-\boldsymbol{x}'|}\,dS'_j - \frac{1}{4\pi}\int_{\mathcal{M}} \partial'^j U'^2 \partial'_j \frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x' \\
&= \frac{1}{4\pi}\oint_{\partial\mathcal{M}} \left(\frac{\partial'^j U'^2}{|\boldsymbol{x}-\boldsymbol{x}'|} - U'^2\partial'_j \frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\right) dS'_j \\
&\quad + \frac{1}{4\pi}\int_{\mathcal{M}} U'^2 \nabla'^2 \frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x' \\
&= -U^2 + \frac{1}{4\pi}\oint_{\partial\mathcal{M}} \left(\frac{\partial'^j U'^2}{|\boldsymbol{x}-\boldsymbol{x}'|} - U'^2\partial'_j \frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\right) dS'_j .
\end{aligned} \tag{7.75}$$

The surface term is evaluated at $r' = \mathcal{R}$, and because U' falls off as $(r')^{-1}$ at large distances from the matter distribution, it makes a contribution that scales as $\mathcal{R}^{-2}$. As with all $\mathcal{R}$-dependent terms in the potentials $h^{\alpha\beta}_{\mathcal{N}}$, we may discard it because it will eventually be cancelled by an equal and opposite term in $h^{\alpha\beta}_{\mathcal{W}}$.

It is interesting to note that if the Poisson integral of $\nabla^2 U^2$ were extended to infinity instead of being truncated to the domain $\mathcal{M}$, it would be exactly equal to $-U^2$. This may seem like a trivial observation, but we wish to call attention to the fact that the solution to the differential equation $\nabla^2 f = \nabla^2 g$ is not necessarily the obvious $f = g$. The actual solution may also include a solution to Laplace's equation $\nabla^2 f = 0$, and the correct mixture of particular and homogeneous solutions depends on the boundary conditions captured by the surface integral in Eq. (7.75). When the boundary conditions are such that the surface integral vanishes except for $\mathcal{R}$-dependent terms, the particular solution $f = g$ is justified. When, however, the surface integral returns contributions that are independent of $\mathcal{R}$, the relevant solution is no longer the simple $f = g$.

We have now taken care of the leading-order, Poisson-integral terms in Eq. (7.15). Proceeding to the next order in $h^{00}_{\mathcal{N}}$, we examine the superpotential term, in which we may insert the leading-order expression $c^{-2}\tau^{00}_1 = \rho^* + O(c^{-2})$, because the correction at order c^{-2} would contribute to $h^{00}_{\mathcal{N}}$ at order c^{-6}. This gives rise to the post-Newtonian superpotential

$$X(t, \boldsymbol{x}) := G \int \rho^*(t, \boldsymbol{x}')|\boldsymbol{x} - \boldsymbol{x}'|\, d^3x' \,, \tag{7.76}$$

in which the integral over $\mathscr{M}$ is naturally truncated to the volume occupied by the matter distribution. With this notation we observe that the superpotential term in $h^{00}_{\mathscr{N}}$ is proportional to $\partial_t^2 X$. Since $\nabla^2|\boldsymbol{x} - \boldsymbol{x}'| = 2|\boldsymbol{x} - \boldsymbol{x}'|^{-1}$ when $\boldsymbol{x} \neq \boldsymbol{x}'$, we see that the superpotential satisfies the Poisson equation

$$\nabla^2 X = 2U \,, \tag{7.77}$$

and X is therefore sourced by the Newtonian potential. The connection between Eqs. (7.76) and (7.77) is further explored in Box 7.3.

Collecting results, we have obtained the following expressions for the scalar potentials $h^{00}_{\mathscr{N}}$ and $h^{kk}_{\mathscr{N}}$:

$$h^{00}_{2\mathscr{N}} = \frac{4}{c^2}U + \frac{1}{c^4}\left(7U^2 + 4\psi - 4V + 2\frac{\partial^2 X}{\partial t^2}\right) - \frac{2G}{3c^5}\dddot{\mathcal{I}}^{kk}(t) + O(c^{-6}) \,, \tag{7.78a}$$

$$h^{kk}_{2\mathscr{N}} = \frac{1}{c^4}\left(U^2 + 4V\right) - 2\frac{G}{c^5}\dddot{\mathcal{I}}^{kk}(t) + O(c^{-6}) \,. \tag{7.78b}$$

These expressions are accurate up to order c^{-6}, and they incorporate Newtonian, 1PN, and 1.5PN terms. Once we have obtained the spacetime metric from the potentials, the terms of order c^{-5} will be shown to represent coordinate artifacts that can be removed by a coordinate transformation.

Box 7.3 Definition of the superpotential

The post-Newtonian superpotential X is defined by Eq. (7.76), and this leads to the Poisson equation displayed in Eq. (7.77). Here we ask whether defining the superpotential through

$$\nabla^2 X = 2U$$

necessarily leads back to the integral representation of Eq. (7.76). We shall see that the answer to this question is subtle, and provides further illustration of the fact that boundary conditions and solutions to Laplace's equation sometimes play an important role in solving Poisson's equation.

The general solution to Poisson's equation for the superpotential is

$$X(t, \boldsymbol{x}) = -\frac{1}{2\pi}\int \frac{U(t, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}\, d^3x' + X_0(t, \boldsymbol{x}),$$

in which X_0 is a solution to $\nabla^2 X_0 = 0$. But the integral is ill defined; because U falls off as $(r')^{-1}$ at large distances, the integrand behaves as $(r')^{-2}$, and since it is multiplied by the integration measure $r'^2\, dr'$, the integral is linearly divergent. To provide a well-defined prescription for the Poisson integral, we truncate the domain of integration to $\mathscr{M}$. This amounts to modifying the Poisson equation to $\nabla^2 X = 2U\Theta(\mathcal{R} - r)$, in which Θ is the Heaviside step function; the modification produces no noticeable changes in the near zone.

Inserting the standard expression of Eq. (7.38) for the Newtonian potential, we find that the superpotential can be expressed as

$$X(t,\boldsymbol{x}) = G\int \rho^*(t,\boldsymbol{y})K(\boldsymbol{x};\boldsymbol{y})\,d^3y + X_0(t,\boldsymbol{x}),$$

in which the two-point function $K(\boldsymbol{x};\boldsymbol{y})$ is defined by

$$K(\boldsymbol{x};\boldsymbol{y}) := -\frac{1}{2\pi}\int_{\mathscr{M}} \frac{d^3x'}{|\boldsymbol{x}-\boldsymbol{x}'||\boldsymbol{x}'-\boldsymbol{y}|}.$$

To evaluate this we exploit the observation that K can depend on $\boldsymbol{x}$ and $\boldsymbol{y}$ only through the combination $\boldsymbol{x}-\boldsymbol{y}$, and thereby set $\boldsymbol{y}=\boldsymbol{0}$ to simplify the integral. Making use of the addition theorem for spherical harmonics, we find that $K(\boldsymbol{x};\boldsymbol{0}) = -2\int_0^{\mathcal{R}}(r_>)^{-1}r'\,dr'$, in which $r_>$ is the greater of r and r'. This returns $r-2\mathcal{R}$, and we conclude that the two-point function is given by

$$K(\boldsymbol{x};\boldsymbol{y}) = |\boldsymbol{x}-\boldsymbol{y}| - 2\mathcal{R}.$$

Inserting this within the integral for the superpotential, we obtain

$$X(t,\boldsymbol{x}) = G\int \rho^*(t,\boldsymbol{x}')|\boldsymbol{x}-\boldsymbol{x}'|\,d^3x' - 2m\mathcal{R} + X_0(t,\boldsymbol{x}),$$

with $m := \int \rho^*(t,\boldsymbol{x}')\,d^3x'$ denoting the total rest-mass of the matter distribution. Choosing $X_0 = 2m\mathcal{R}$, we reproduce the original definition of the superpotential.

It is interesting to note that since it is $\partial_t^2 X$ that appears in the gravitational potentials, the addition of $-2m\mathcal{R} + X_0$ to the integral is immaterial, so long as X_0 does not depend on time. The superpotential, therefore, is sufficiently robust to withstand the ambiguities associated with the choice of solution to $\nabla^2 X = 2U$.

Vector class

For our purposes it is necessary to evaluate the potential $h^{0j}_{\mathcal{N}}$ to order c^{-3} only. Our expression for $c^{-1}\tau_1^{0j}$ in Eq. (7.54b) is more accurate than we need, and we may truncate it to its leading term $\rho^* v^j + O(c^{-2})$. Consulting Eq. (7.15b), we see that the leading term in the potential is given by a Poisson integral constructed from $c^{-1}\tau_1^{0j}$, and we obtain

$$h^{0j}_{2\mathcal{N}} = \frac{4}{c^3}U^j + O(c^{-5}), \tag{7.79}$$

where U^j is the vector potential defined by Eq. (7.39). In principle we have enough information to calculate the correction terms at order c^{-5}, but these will not be needed in our future developments.

Tensor class

The computation of $h^{jk}_{\mathcal{N}}$ is more involved, because its source term contains a field contribution that is not as easy to deal with as it was with the scalar potentials. Returning to Eq. (7.54) and exploiting once more the identity of Eq. (7.70), we express τ_1^{jk} in the form

$$\tau_1^{jk} = \rho^*\left(v^j v^k - \frac{1}{2}U\delta^{jk}\right) + p\,\delta^{jk} - \frac{1}{16\pi G}\delta^{jk}\nabla^2 U^2 + \frac{1}{4\pi G}\partial^j U \partial^k U + O(c^{-2})\,. \tag{7.80}$$

Consulting Eq. (7.15c), we see that the leading term in the potential is a Poisson integral constructed from τ_1^{jk}. The first three terms have compact support, and they give rise to the tensorial potential

$$W^{jk}(t,\boldsymbol{x}) := G\int \frac{\rho^{*\prime}\left(v'^j v'^k - \frac{1}{2}U'\delta^{jk}\right) + p'\delta^{jk}}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x'\,, \tag{7.81}$$

which satisfies the Poisson equation

$$\nabla^2 W^{jk} = -4\pi G\left(\rho^* v^j v^k - \frac{1}{2}\rho^* U\delta^{jk} + p\,\delta^{jk}\right). \tag{7.82}$$

The fourth term involves $\nabla^2 U^2$, which we know how to handle, and which produces a contribution proportional to $\delta^{jk}U^2$ to $h^{jk}_{\mathcal{N}}$. The fifth and final term is the hard one. To account for it we introduce another tensorial potential defined by

$$\chi^{jk}(t,\boldsymbol{x}) := \frac{1}{4\pi}\int_{\mathscr{M}} \frac{\partial^{j'}U'\partial^{k'}U'}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x'\,, \tag{7.83}$$

which satisfies the Poisson equation

$$\nabla^2\chi^{jk} = -\partial^j U\partial^k U\,. \tag{7.84}$$

Because the Poisson integral in Eq. (7.83) is truncated at $r' = \mathcal{R}$, the source term on the right-hand side of the Poisson equation should be multiplied by $\Theta(\mathcal{R}-r)$, as was discussed in Box 7.3. But since the truncation produces no noticeable changes within the near zone, we have kept it implicit in Eq. (7.84).

Armed with these tensorial potentials, we find that the gravitational potentials can be expressed as

$$h^{jk}_{2\mathcal{N}} = \frac{1}{c^4}\left(4W^{jk} + U^2\delta^{jk} + 4\chi^{jk}\right) - 2\frac{G}{c^5}\dddot{\mathcal{I}}^{jk}(t) + O(c^{-6})\,, \tag{7.85}$$

where we have included the $O(c^{-5})$ term for completeness.

Computation of χ^{jk}

We must now face the computation of χ^{jk}, as defined by Eq. (7.83). Returning to the standard expression of Eq. (7.38) for the Newtonian potential, we differentiate it and obtain

$$\begin{aligned}\partial_{j'}U' &= G\int d^3y_1\,\rho_1^*\,\frac{\partial}{\partial x'^j}\frac{1}{|\boldsymbol{x}'-\boldsymbol{y}_1|}\\ &= -G\int d^3y_1\,\rho_1^*\,\frac{\partial}{\partial y_1^j}\frac{1}{|\boldsymbol{x}'-\boldsymbol{y}_1|}\,,\end{aligned} \tag{7.86}$$

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in which $\boldsymbol{y}_1$ is an integration variable, and $\rho_1^* := \rho^*(t, \boldsymbol{y}_1)$. Expressing $\partial_{k'} U'$ in a similar way, in terms of an independent integration variable $\boldsymbol{y}_2$, and inserting these expressions in the Poisson integral for χ^{jk}, we arrive at

$$\chi^{jk} = G^2 \int d^3y_1 d^3y_2\, \rho_1^* \rho_2^* \frac{\partial^2}{\partial y_1^j \partial y_2^k} K(\boldsymbol{x}; \boldsymbol{y}_1, \boldsymbol{y}_2), \tag{7.87}$$

where

$$K(\boldsymbol{x}; \boldsymbol{y}_1, \boldsymbol{y}_2) := \frac{1}{4\pi} \int_{\mathscr{M}} \frac{d^3x'}{|\boldsymbol{x} - \boldsymbol{x}'||\boldsymbol{x}' - \boldsymbol{y}_1||\boldsymbol{x}' - \boldsymbol{y}_2|} \tag{7.88}$$

is a three-point function that must now be evaluated. This computation is presented in Box 7.4, and the end result is

$$K(\boldsymbol{x}; \boldsymbol{y}_1, \boldsymbol{y}_2) = 1 - \ln \frac{S}{2\mathcal{R}}, \tag{7.89}$$

where

$$S := r_1 + r_2 + r_{12}, \tag{7.90}$$

with the notations

$$r_1 := |\boldsymbol{x} - \boldsymbol{y}_1|, \qquad r_2 := |\boldsymbol{x} - \boldsymbol{y}_2|, \qquad r_{12} := |\boldsymbol{y}_1 - \boldsymbol{y}_2|. \tag{7.91}$$

We also introduce the corresponding separation vectors

$$\boldsymbol{r}_1 := \boldsymbol{x} - \boldsymbol{y}_1, \qquad \boldsymbol{r}_2 := \boldsymbol{x} - \boldsymbol{y}_2, \qquad \boldsymbol{r}_{12} := \boldsymbol{y}_1 - \boldsymbol{y}_2, \tag{7.92}$$

and the unit vectors

$$\boldsymbol{n}_1 := \frac{\boldsymbol{r}_1}{r_1}, \qquad \boldsymbol{n}_2 := \frac{\boldsymbol{r}_2}{r_2}, \qquad \boldsymbol{n}_{12} := \frac{\boldsymbol{r}_{12}}{r_{12}}. \tag{7.93}$$

The dependence of K on $\mathcal{R}$ comes from the fact that the domain of integration is truncated at $r' = \mathcal{R}$. This dependence plays no role, however, because K is differentiated as soon as it is inserted within Eq. (7.87). These derivatives are straightforward to compute, and we obtain

$$\frac{\partial^2 K}{\partial y_1^j \partial y_2^k} = \frac{\left(n_1^j - n_{12}^j\right)\left(n_2^k + n_{12}^k\right)}{S^2} - \frac{n_{12}^j n_{12}^k - \delta^{jk}}{S r_{12}}. \tag{7.94}$$

We then arrive at

$$\begin{aligned} \chi^{jk} = G^2 &\int \frac{\rho_1^* \rho_2^* \left(n_1^j - n_{12}^j\right)\left(n_2^k + n_{12}^k\right)}{S^2}\, d^3y_1 d^3y_2 \\ &- G^2 \int \frac{\rho_1^* \rho_2^* \left(n_{12}^j n_{12}^k - \delta^{jk}\right)}{S r_{12}}\, d^3y_1 d^3y_2. \end{aligned} \tag{7.95}$$

It is easy to check that each integral is symmetric in the jk indices; this property is evident in the second integral, and to establish it for the first it is necessary to swap the variables of integration, $\boldsymbol{y}_1 \leftrightarrow \boldsymbol{y}_2$, keeping in mind that $\boldsymbol{n}_{12} \to \boldsymbol{n}_{21} = -\boldsymbol{n}_{12}$ under this operation.

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Note that the trace $\chi := \delta_{jk}\chi^{jk}$ is given by the Poisson potential of $\frac{1}{2}\partial_j U \partial^j U$. Using the identity of Eq. (7.70), it is easy to see that χ can be expressed as

$$\chi = -\frac{1}{2}U^2 + G\int_{\mathscr{M}} \frac{\rho'^* U'}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3x' . \tag{7.96}$$

By inserting the Poisson integral for U, we can express this in the form

$$\chi = -\frac{1}{2}G^2 \int \frac{\rho_1^*\rho_2^*\, d^3y_1 d^3y_2}{|\boldsymbol{x}-\boldsymbol{y}_1||\boldsymbol{x}-\boldsymbol{y}_2|} + G^2 \int \frac{\rho_1^*\rho_2^*\, d^3y_1 d^3y_2}{|\boldsymbol{x}-\boldsymbol{y}_1||\boldsymbol{y}_1-\boldsymbol{y}_2|} . \tag{7.97}$$

The second integral can be written in the symmetric form

$$\frac{1}{2}\int \frac{\rho_1^*\rho_2^*\, d^3y_1 d^3y_2}{|\boldsymbol{x}-\boldsymbol{y}_1||\boldsymbol{y}_1-\boldsymbol{y}_2|} + \frac{1}{2}\int \frac{\rho_1^*\rho_2^*\, d^3y_1 d^3y_2}{|\boldsymbol{x}-\boldsymbol{y}_2||\boldsymbol{y}_1-\boldsymbol{y}_2|},$$

and this gives

$$\chi = \frac{1}{2}G^2 \int \rho_1^*\rho_2^* \biggl(-\frac{1}{r_1 r_2} + \frac{1}{r_1 r_{12}} + \frac{1}{r_2 r_{12}} \biggr) d^3y_1 d^3y_2 . \tag{7.98}$$

Our final expression is

$$\chi = \frac{1}{2}G^2 \int \rho_1^*\rho_2^* \frac{r_1 + r_2 - r_{12}}{r_1 r_2 r_{12}}\, d^3y_1 d^3y_2 , \tag{7.99}$$

and we may check that the trace of Eq. (7.95) reproduces this. The calculation is aided by the identities

$$\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = \frac{r_1^2 + r_2^2 - r_{12}^2}{2r_1 r_2}, \tag{7.100a}$$

$$\boldsymbol{n}_1 \cdot \boldsymbol{n}_{12} = \frac{r_2^2 - r_1^2 - r_{12}^2}{2r_1 r_{12}}, \tag{7.100b}$$

$$\boldsymbol{n}_2 \cdot \boldsymbol{n}_{12} = \frac{r_2^2 - r_1^2 + r_{12}^2}{2r_2 r_{12}} , \tag{7.100c}$$

involving the unit vectors defined by Eq. (7.93).

Box 7.4 **Three-point function $K(\boldsymbol{x}; \boldsymbol{y}_1, \boldsymbol{y}_2)$**

The computation of the three-point function defined by Eq. (7.88) follows some of the same steps that were used to calculate the two-point function $K(\boldsymbol{x}; \boldsymbol{y})$ in Box 7.3.

We note first that $K(\boldsymbol{x}; \boldsymbol{y}_1, \boldsymbol{y}_2)$ satisfies

$$\nabla^2 K(\boldsymbol{x}; \boldsymbol{y}_1, \boldsymbol{y}_2) = -\frac{1}{|\boldsymbol{x}-\boldsymbol{y}_1||\boldsymbol{x}-\boldsymbol{y}_2|}, \tag{1}$$

and verify that $K_\mathrm{p} = -\ln S$ is a particular solution. The relation implies that

$$\nabla^2 K_\mathrm{p} = -\frac{1}{S^2}(S\nabla^2 S - \partial_j S \partial^j S),$$

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and the various derivatives can be computed from the definition of S provided in Eq. (7.90). We have, for example, $\partial^j S = n_1^j + n_2^j$, from which it follows that

$$\partial^{jk} S = -\frac{n_1^j n_1^k - \delta^{jk}}{r_1} - \frac{n_2^j n_2^k - \delta^{jk}}{r_2}.$$

From this, and the helpful identities of Eqs. (7.100), we obtain

$$\nabla^2 S = 2\frac{r_1 + r_2}{r_1 r_2}, \qquad \partial_j S \partial^j S = \frac{(r_1 + r_2 - r_{12})S}{r_1 r_2}.$$

Collecting results, we confirm that $K_{\rm p}$ is a solution to $\nabla^2 K = -1/(r_1 r_2)$.

To this we must add a suitable solution $K_{\rm h}$ to Laplace's equation. The solution to the homogeneous equation must be non-singular in all three variables $\boldsymbol{x}$, $\boldsymbol{y}_1$, and $\boldsymbol{y}_2$, because the singularity structure required by Eq. (1) is already contained in $K_{\rm p}$. Furthermore, $K_{\rm h}$ must be dimensionless, and the only possibility is to make it equal to a constant. We are therefore looking for a solution of the form

$$K = K_0 - \ln(r_1 + r_2 + r_{12}),$$

where K_0 is a dimensionless constant. To determine this we carry out an independent computation of the special value $K(\boldsymbol{x}; \boldsymbol{0}, \boldsymbol{0})$, and compare our result to $K_0 - \ln(2r)$, which follows from the general expression.

From Eq. (7.88) we have

$$K(\boldsymbol{x}; \boldsymbol{0}, \boldsymbol{0}) = \frac{1}{4\pi}\int_{\mathscr{M}} \frac{d^3x'}{|\boldsymbol{x} - \boldsymbol{x}'||\boldsymbol{x}'|^2} = \frac{1}{4\pi}\int_0^{\mathcal{R}} \frac{dr'd\Omega'}{|\boldsymbol{x} - \boldsymbol{x}'|}.$$

Invoking the addition theorem for spherical harmonics, this is simply $\int_0^{\mathcal{R}} (r_>)^{-1}\, dr'$, and evaluating the integral gives $K(\boldsymbol{x}; \boldsymbol{0}, \boldsymbol{0}) = 1 + \ln(\mathcal{R}/r)$. This allows us to conclude that $K_0 = 1 + \ln(2\mathcal{R})$. Collecting results, we obtain the expression displayed in Eq. (7.89).

7.3.4 Wave-zone contribution to potentials

In this subsection we estimate $h^{\alpha\beta}_{\mathscr{W}}$, the wave-zone contribution to the second-iterated potentials, still assuming that the field point $\boldsymbol{x}$ is within the near zone. Techniques to carry out such a computation were described back in Sec. 6.3.5, and crude estimates were obtained toward the end of that section. These ignore numerical factors and terms that depend explicitly on $\mathcal{R}$, but they are sufficient to allow us to conclude that

$$h^{00}_{\mathscr{W}} = O(c^{-8}), \qquad h^{0j}_{\mathscr{W}} = O(c^{-8}), \qquad h^{jk}_{\mathscr{W}} = O(c^{-8}). \tag{7.101}$$

This is far beyond the 1PN accuracy of our calculations in this section, and we shall therefore ignore the wave-zone contribution to $h^{\alpha\beta}_2$.

To reach this conclusion we refer to Eq. (6.105), which applies to source terms of the form displayed in Eq. (6.98). In our current application, $\tau^{\alpha\beta}_1$ is built entirely from $(-g)t^{\alpha\beta}_{\rm LL}$

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CUUK2552-07 CUUK2552-Poisson 978 1 107 03286 6 December 16, 2013 10:51

as displayed in Eqs. (7.48), by inserting the wave-zone potentials h_1^{00} and h_1^{0j} given by Eqs. (7.32). Focusing our attention on τ_1^{00} for concreteness, and ignoring all numerical and angle-dependent factors, we find that it has a structure given schematically by

$$\frac{G^2}{c^4}\left[\frac{M_0^2}{r^4}+\frac{M_0\mathcal{I}^{jk}}{r^6}+\frac{M_0\dot{\mathcal{I}}^{jk}}{cr^5}+\frac{M_0\ddot{\mathcal{I}}^{jk}}{c^2r^4}+\frac{M_0\dddot{\mathcal{I}}^{jk}}{c^3r^3}+\cdots\right], \tag{7.102}$$

in which the ellipsis designates terms of higher post-Newtonian order. Each term is of the form $f(\tau)/r^n$ required for the integration techniques of Sec. 6.3.5. Ignoring the overall factor of G^2/c^4, we see, for example, that for $n=3$ we have $f = M_0\dddot{\mathcal{I}}^{jk}/c^3$, and that for $n=4$ we have $f = M_0^2 + M_0\ddot{\mathcal{I}}^{jk}/c^2$. According to Eq. (6.109), an estimate of $h_{\mathscr{W}}^{00}$ for each contributing n is $c^{-(n-2)}f^{(n-2)} + c^{-(n-1)}rf^{(n-1)}$. The dominant term in a post-Newtonian expansion is $c^{-(n-2)}f^{(n-2)}$, and restoring the factor of G^2/c^4, we find that for each n, $h_{\mathscr{W}}^{00}$ is estimated as

$$\frac{G^2M_0}{c^8}\frac{d^4\mathcal{I}^{jk}}{d\tau^4}. \tag{7.103}$$

This is of order c^{-8}, and contributes to h_2^{00} at 3PN order. A similar result follows for the other components of $h_{\mathscr{W}}^{\alpha\beta}$, and we arrive at the statement of Eq. (7.101).

In fact, a detailed computation shows that these contributions are actually gauge artifacts that can be removed by a suitable coordinate transformation. The first instance in which $h_{\mathscr{W}}^{\alpha\beta}$ makes a non-trivial contribution to the near-zone potentials turns out to be at 4PN order. In any event, we see that $h_{\mathscr{W}}^{\alpha\beta}$ is far too small to contribute to our 1PN potentials, and for this reason we do not need to calculate it in detail.

7.3.5 Near-zone potentials: Final answer

We are now ready to collect our results and display the final expression for the second-iterated potentials $h_2^{\alpha\beta}$ in the near zone. Our results are summarized in Box 7.5.

Box 7.5 **Near-zone potentials**

Combining Eqs. (7.78), (7.79), (7.85), and (7.101), we find that the near-zone gravitational potentials are given by

$$\begin{aligned}
h_2^{00} &= \frac{4}{c^2}U + \frac{1}{c^4}\left(7U^2 + 4\psi - 4V + 2\frac{\partial^2 X}{\partial t^2}\right) - \frac{2G}{3c^5}\dddot{\mathcal{I}}^{kk}(t) + O(c^{-6}),\\
h_2^{0j} &= \frac{4}{c^3}U^j + O(c^{-5}),\\
h_2^{jk} &= \frac{1}{c^4}\left(4W^{jk} + U^2\delta^{jk} + 4\chi^{jk}\right) - 2\frac{G}{c^5}\dddot{\mathcal{I}}^{jk}(t) + O(c^{-6}),\\
h_2^{kk} &= \frac{1}{c^4}\left(U^2 + 4V\right) - \frac{2G}{c^5}\dddot{\mathcal{I}}^{kk}(t) + O(c^{-6}).
\end{aligned}$$

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The potentials that make up $h^{\alpha\beta}$ satisfy the Poisson equations

$$\begin{aligned}
\nabla^2 U &= -4\pi G\rho^* , \\
\nabla^2 \psi &= -4\pi G\rho^* \left(\frac{3}{2}v^2 - U + \Pi + 3p/\rho^*\right), \\
\nabla^2 V &= -4\pi G\rho^* \left(v^2 - \frac{1}{2}U + 3p/\rho^*\right), \\
\nabla^2 X &= 2U , \\
\nabla^2 U^j &= -4\pi G\rho^* v^j , \\
\nabla^2 W^{jk} &= -4\pi G\left(\rho^* v^j v^k - \frac{1}{2}\rho^* U\delta^{jk} + p\delta^{jk}\right), \\
\nabla^2 \chi^{jk} &= -\partial^j U \partial^k U .
\end{aligned}$$

The solutions are

$$\begin{aligned}
U &= G\int \frac{\rho^{*\prime}}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x' , \\
\psi &= G\int \frac{\rho^{*\prime}\left(\frac{3}{2}v'^2 - U' + \Pi'\right) + 3p'}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x' , \\
V &= G\int \frac{\rho^{*\prime}\left(v'^2 - \frac{1}{2}U'\right) + 3p'}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3x' , \\
X &= G\int \rho^{*\prime}|\boldsymbol{x}-\boldsymbol{x}'|\, d^3x' , \\
U^j &= G\int \frac{\rho^{*\prime}v'^j}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x' , \\
W^{jk} &= G\int \frac{\rho^{*\prime}\left(v'^j v'^k - \frac{1}{2}U'\delta^{jk}\right) + p'\delta^{jk}}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x' , \\
\chi^{jk} &= G^2 \int \frac{\rho_1^*\rho_2^*\left(n_1^j - n_{12}^j\right)\left(n_2^k + n_{12}^k\right)}{S^2}\, d^3y_1 d^3y_2 \\
&\quad - G^2 \int \frac{\rho_1^*\rho_2^*\left(n_{12}^j n_{12}^k - \delta^{jk}\right)}{S r_{12}}\, d^3y_1 d^3y_2 .
\end{aligned}$$

The potentials are evaluated at time t and position $\boldsymbol{x}$; the sources are evaluated at the same time but at position $\boldsymbol{x}'$. We use the notation $\boldsymbol{r}_1 := \boldsymbol{x} - \boldsymbol{y}_1, r_1 := |\boldsymbol{r}_1|, \boldsymbol{n}_1 := \boldsymbol{r}_1/r_1$ (and similarly for $\boldsymbol{r}_2, r_2$, and $\boldsymbol{n}_2$), as well as $\boldsymbol{r}_{12} := \boldsymbol{y}_1 - \boldsymbol{y}_2, r_{12} := |\boldsymbol{r}_{12}|$, and $\boldsymbol{n}_{12} := \boldsymbol{r}_{12}/r_{12}$, in which $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ are integration variables. We also have $S := r_1 + r_2 + r_{12}$, and the trace of χ^{jk} is given by

$$\chi = -\frac{1}{2}U^2 + G\int_{\mathscr{M}} \frac{\rho^{*\prime}U'}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3x' .$$

From Eq. (7.23) we find that the potentials give rise to the spacetime metric

$$g_{00} = -1 + \frac{2}{c^2}U + \frac{2}{c^4}\left(\psi - U^2 + \frac{1}{2}\frac{\partial^2 X}{\partial t^2}\right) - \frac{4G}{3c^5}\dddot{\mathcal{I}}^{kk}(t) + O(c^{-6}), \tag{7.104a}$$

$$g_{0j} = -\frac{4}{c^3}U_j + O(c^{-5}), \tag{7.104b}$$

$$g_{jk} = \delta_{jk}\left[1 + \frac{2}{c^2}U + \frac{2}{c^4}\left(\psi + U^2 - 2V + \frac{1}{2}\frac{\partial^2 X}{\partial t^2}\right)\right] + \frac{4}{c^4}\left(W^{jk} + \chi^{jk}\right) - 2\frac{G}{c^5}\dddot{\mathcal{I}}^{\langle jk\rangle}(t) + O(c^{-6}). \tag{7.104c}$$

This metric is too accurate for most of our purposes. As we indicated back in Sec. 7.1.3, in order to describe the slow motion of a weakly gravitating system accurately through 1PN order, we require g_{00} to order c^{-4}, g_{0j} to order c^{-3}, and g_{jk} to order c^{-2}. For this application our previous expressions can therefore be truncated to

$$g_{00} = -1 + \frac{2}{c^2}U + \frac{2}{c^4}\left(\psi - U^2 + \frac{1}{2}\frac{\partial^2 X}{\partial t^2}\right) + O(c^{-5}), \tag{7.105a}$$

$$g_{0j} = -\frac{4}{c^3}U_j + O(c^{-5}), \tag{7.105b}$$

$$g_{jk} = \delta_{jk}\left(1 + \frac{2}{c^2}U\right) + O(c^{-4}). \tag{7.105c}$$

This metric forms the basis of what is known as *post-Newtonian theory*. Chapters 8 through 10 will be devoted to the details and many applications of this approximation to general relativity.

We have previously indicated that the c^{-5} term in g_{00} is a coordinate artifact that has no impact on the physics of our gravitating system. Because it depends only on time, this term may in fact be removed by a transformation of the time coordinate given by

$$t = t' - \frac{2G}{3c^5}\ddot{\mathcal{I}}^{kk}(t') + O(c^{-7}). \tag{7.106}$$

It is a simple exercise to show that the time-time component of the transformed metric, expressed in terms of the new time t', no longer contains a term at order c^{-5}; the other components of the metric are not affected by the transformation. It should be noted that the transformed coordinates are no longer harmonic; the c^{-5} term must stay if we insist on using harmonic coordinates. A more careful calculation reveals that the transformation generates non-trivial terms in g_{00} at order c^{-7}, or at 2.5PN order; these must then be combined with other 2.5PN terms in order to give a correct description of radiation-reaction effects. We return to this theme in Chapter 12.

Box 7.6 **Post-Minkowskian theory and the slow-motion approximation**

The advantages of incorporating a slow-motion condition within post-Minkowskian theory should be pretty clear by now, quite apart from the physical relevance of slow motion within a weak-field context. Had we not expanded the various retarded potentials in powers of c^{-1} right from the start, we would have been faced

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with the need to evaluate fully retarded potentials such as

$$\int \frac{\rho^*(t-|\boldsymbol{x}-\boldsymbol{x}'|,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x' ,$$

$$\int \frac{\rho^*(t-|\boldsymbol{x}-\boldsymbol{x}'|,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\frac{\rho^*(t-|\boldsymbol{x}-\boldsymbol{x}'|-|\boldsymbol{x}'-\boldsymbol{x}''|,\boldsymbol{x}'')}{|\boldsymbol{x}'-\boldsymbol{x}''|}\,d^3x'd^3x'' ,$$

$$\int_{\mathscr{M}} \frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\partial'_j\frac{\rho^*(t-|\boldsymbol{x}-\boldsymbol{x}'|-|\boldsymbol{x}'-\boldsymbol{x}''|,\boldsymbol{x}'')}{|\boldsymbol{x}'-\boldsymbol{x}''|}
\times \partial'_k\frac{\rho^*(t-|\boldsymbol{x}-\boldsymbol{x}'|-|\boldsymbol{x}'-\boldsymbol{x}'''|,\boldsymbol{x}''')}{|\boldsymbol{x}'-\boldsymbol{x}'''|}\,d^3x'd^3x''d^3x''' ;$$

these expressions are the fully retarded analogues of U, $\int \rho'^*U'|\boldsymbol{x}-\boldsymbol{x}'|^{-1}d^3x'$, and χ^{jk}, respectively.

Such potentials lead to hopeless complications. Even a relatively simple potential, such as the first one listed above, leads to difficult computations because of the need to account for the retardation condition. Examples of such complexities are known in Maxwell's theory, in which the evaluation of the retarded potential is difficult even for the simple case of a single point charge (remember the Liénard–Wiechert potentials?). The nonlinear potentials are even more challenging, as they involve nested retardation conditions; such potentials do not occur in electromagnetism, because of the linearity of Maxwell's equations.

In the early 1960s, Peter Havas and Joshua Goldberg, together with their students and collaborators, worked on post-Minkowskian theory in order to study gravitational radiation, but they chose *not* to incorporate the slow-motion condition. Very quickly they ran into the difficulties noted above, and as a result, they were unable to go beyond the first iteration of the relaxed field equations. And even for the first-iterated potentials, they were able to evaluate quantities like the retarded Newtonian potential only for specific motions, such as circular orbits, where mathematical techniques from electrodynamics were available. In the 1970s, Havas's student Arnold Rosenblum worked on obtaining the second iteration, but progress was extremely slow, and his untimely death in 1991 essentially brought this program to an end without any definitive conclusion.

7.4 Second iteration: Wave zone

Our final task in this chapter is to obtain expressions for the second-iterated potentials when the field point $\boldsymbol{x}$ is in the wave zone, where $r := |\boldsymbol{x}| > \mathcal{R}$.

7.4.1 Near-zone contribution to potentials

Equations (7.32) give us formal expressions for the potentials $h^{\alpha\beta}_{\mathscr{N}}$ evaluated in the wave zone. Recalling our discussion of Sec. 7.2.3, in which we observed that each successive multipole moment brings an additional factor of v_c/c to the post-Newtonian ordering,

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we have that

$$h^{00}_{\mathcal{N}} = \underbrace{\frac{4GM}{c^2 r}}_{0+1\text{PN}} + \underbrace{\frac{2G}{c^2}\partial_{jk}\left[\frac{\mathcal{I}^{jk}(\tau)}{r}\right]}_{1\text{PN}} - \underbrace{\frac{2G}{3c^2}\partial_{jkn}\left[\frac{\mathcal{I}^{jkn}(\tau)}{r}\right]}_{1.5\text{PN}} + \cdots\,, \tag{7.107a}$$

$$\begin{aligned} h^{0j}_{\mathcal{N}} &= -\underbrace{\frac{2G}{c^3}\frac{(\boldsymbol{n}\times\boldsymbol{J})^j}{r^2}}_{1\text{PN}} - \underbrace{\frac{2G}{c^3}\partial_k\left[\frac{\dot{\mathcal{I}}^{jk}(\tau)}{r}\right]}_{1\text{PN}} \\ &\quad \underbrace{- \frac{G}{3c^3}\partial_{kn}\left[\frac{\dot{\mathcal{I}}^{jkn}(\tau) - 2\epsilon^{mjk}\mathcal{J}^{mn}(\tau)}{r}\right]}_{1.5\text{PN}} + \cdots\,, \end{aligned} \tag{7.107b}$$

$$h^{jk}_{\mathcal{N}} = \underbrace{\frac{2G}{c^4}\frac{\ddot{\mathcal{I}}^{jk}(\tau)}{r}}_{1\text{PN}} - \underbrace{\frac{2G}{3c^4}\partial_n\left[\frac{\ddot{\mathcal{I}}^{jkn}(\tau) + 4\epsilon^{mn(j}\dot{\mathcal{J}}^{m|k)}(\tau)}{r}\right]}_{1.5\text{PN}} + \cdots\,, \tag{7.107c}$$

is a post-Newtonian expansion of the potentials that is accurate through 1.5PN order. Recalling Eq. (7.67), we have replaced the monopole moment $\mathcal{I} = M_0$ – the near-zone mass – that originally appeared in Eq. (7.32) with the total mass M, since they agree to order c^{-4}. We recall that M is given by Eq. (7.63), so that it contains both a 0PN rest-mass contribution and 1PN corrections provided by the system's total energy. We have also replaced the near-zone angular momentum $\boldsymbol{J}_0$ by the total angular momentum $\boldsymbol{J}$, since these quantities agree to order c^{-2}.

The multipole moments that appear in Eqs. (7.107) are all functions of retarded time $\tau = t - r/c$. Formally they must be evaluated using the first-iterated forms $\tau_1^{\alpha\beta}$ for the energy-momentum pseudotensor, but since the multipole moments occur at 1PN and 1.5PN orders in the potentials, we may truncate $\tau_1^{\alpha\beta}$ to its leading-order expression $c^{-2}\tau^{00} = \rho^* + O(c^{-2})$ and $c^{-1}\tau^{0j} = \rho^* v^j + O(c^{-2})$. The multipole moments then take the explicit forms

$$\mathcal{I}^{jk}(\tau) = \int \rho^* x^j x^k \, d^3x + O(c^{-2})\,, \tag{7.108a}$$

$$\mathcal{I}^{jkn}(\tau) = \int \rho^* x^j x^k x^n \, d^3x + O(c^{-2})\,, \tag{7.108b}$$

$$\mathcal{J}^{jk}(\tau) = \epsilon^{jab}\int \rho^* v^a x^b x^k \, d^3x + O(c^{-2})\,. \tag{7.108c}$$

With these, our expressions for $h^{\alpha\beta}_{\mathcal{N}}$ are complete.

7.4.2 Wave-zone contribution to potentials

We turn next to the computation of $h^{\alpha\beta}_{\mathcal{W}}$ in the wave zone. To carry this out we insert the first-iterated potentials obtained in Sec. 7.2.3 within $\tau_1^{\alpha\beta}$, and solve the relaxed field equations for the second-iterated potentials. By virtue of Eq. (7.52), only the

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Landau–Lifshitz pseudotensor of Eq. (7.48) makes a contribution to $\tau_1^{\alpha\beta}$. And by virtue of our requirement of 1.5PN overall accuracy for the potentials, we find that the only relevant piece of the first-iterated potentials is the Newtonian term in $h^{00}_{\mathscr{W}}$, given by

$$h^{00}_{\mathscr{W}} = \frac{4GM}{c^2 r} + O(c^{-4}) . \tag{7.109}$$

Inserting this within Eq. (7.48), we find that the components of the energy-momentum pseudotensor are

$$\tau_1^{00} = -\frac{7GM^2}{8\pi r^4} + O(c^{-2}) , \tag{7.110a}$$

$$\tau_1^{0j} = O(c^{-3}) , \tag{7.110b}$$

$$\tau_1^{jk} = \frac{GM^2}{4\pi r^4}\left(n^j n^k - \frac{1}{2}\delta^{jk}\right) , \tag{7.110c}$$

in which $n^j := x^j/r$.

To obtain $h^{\alpha\beta}_{\mathscr{W}}$ we rely on the methods of Sec. 6.3.5, which work for source terms of the form displayed in Eq. (6.98). Our first task is to decompose the effective stress tensor of Eq. (7.110c) in terms of STF angular tensors, refer to Sec. 1.5.3. We invoke the identity $n^j n^k = n^{\langle jk\rangle} + \frac{1}{3}\delta^{jk}$ and rewrite Eq. (7.110c) as

$$\tau_1^{jk} = \frac{G}{4\pi}\frac{M^2}{r^4}\left(n^{\langle jk\rangle} - \frac{1}{6}\delta^{jk}\right) . \tag{7.111}$$

This and Eq. (7.110a) are now of the form of Eq. (7.19), and we identify $f^{00}_{\ell=0}$ with $-\frac{7}{2}GM^2$, $f^{jk}_{\ell=2}$ with GM^2, and $f^{jk}_{\ell=0}$ with $-\frac{1}{6}GM^2\delta^{jk}$. In each case we have that $n = 4$.

The contribution to $h^{\alpha\beta}_{\mathscr{W}}$ from each value of ℓ is given by Eq. (6.105), which we copy here as

$$h^{\alpha\beta}_{\mathscr{W}}(t, \boldsymbol{x}) = \frac{4G}{c^4}\frac{n^{\langle L\rangle}}{r}\left\{\int_0^{\mathcal{R}} ds\, f^{\alpha\beta}(\tau - 2s/c)A(s,r) + \int_{\mathcal{R}}^{\infty} ds\, f^{\alpha\beta}(\tau - 2s/c)B(s,r)\right\} , \tag{7.112}$$

in which $A(s,r) = \int_{\mathcal{R}}^{r+s} P_\ell(\xi)p^{-(n-1)}\,dp$, $B(s,r) = \int_s^{r+s} P_\ell(\xi)p^{-(n-1)}\,dp$, and $\xi = (r + 2s)/r - 2s(r+s)/(rp)$. Because $f^{\alpha\beta}$ is a constant, it can be taken outside of each integral, and the remaining computations are simple. For $\ell = 0$ we get

$$h^{00}_{\mathscr{W}} = 7\left(\frac{GM}{c^2 r}\right)^2\left(1 - 2\frac{r}{\mathcal{R}}\right) , \tag{7.113a}$$

$$h^{jk}_{\mathscr{W}} = \frac{1}{3}\left(\frac{GM}{c^2 r}\right)^2 \delta^{jk}\left(1 - 2\frac{r}{\mathcal{R}}\right), \tag{7.113b}$$

and for $\ell = 2$

$$h^{jk}_{\mathscr{W}} = \left(\frac{GM}{c^2 r}\right)^2 n^{\langle jk\rangle}\left(1 - \frac{4\mathcal{R}}{5r}\right) . \tag{7.114}$$

Discarding all terms involving $\mathcal{R}$, as we are free to do, and adding the results, we arrive at

$$h^{00}_{\mathscr{W}} = 7\left(\frac{GM}{c^2 r}\right)^2, \tag{7.115a}$$

$$h^{jk}_{\mathscr{W}} = \left(\frac{GM}{c^2 r}\right)^2 n^j n^k . \tag{7.115b}$$

The post-Newtonian order of these contributions to h^{00} and h^{jk} is 1.5PN. To see this, we divide each of these expressions by $h^{00} \sim GM/(c^2 r)$ to obtain something proportional to $GM/(c^2 r)$. We next incorporate the fact that the Newtonian acceleration GM/r_c^2 is of order r_c/t_c^2, which makes GM of order r_c^3/t_c^2. Setting $r \sim \lambda_c = ct_c$, we finally get $h^{\alpha\beta}_{\mathscr{W}}/h^{00} \sim r_c^3/(c^3 t_c^3) = (v_c/c)^3$, and conclude that Eqs. (7.115) do indeed make contributions of 1.5PN order to the gravitational potentials.

We pull everything together and summarize our results in Box 7.7. It is instructive to note that in the limit of a static, spherically symmetric body, the results correspond precisely to the post-Newtonian expansion of the Schwarzschild metric. This statement is established in Exercise 7.7.

Box 7.7 Wave-zone fields

Combining Eqs. (7.107) and (7.115), we find that the wave-zone gravitational potentials are given by

$$h^{00} = \frac{4G}{c^2}\left[\frac{M}{r} + \frac{1}{2}\partial_{jk}\left(\frac{\mathcal{I}^{jk}}{r}\right) - \frac{1}{6}\partial_{jkn}\left(\frac{\mathcal{I}^{jkn}}{r}\right) + \frac{7}{4}\frac{GM^2}{c^2 r^2} + \cdots\right],$$

$$h^{0j} = \frac{4G}{c^3}\left[-\frac{1}{2}\frac{(\boldsymbol{n}\times\boldsymbol{J})^j}{r^2} - \frac{1}{2}\partial_k\left(\frac{\dot{\mathcal{I}}^{jk}}{r}\right) - \frac{1}{12}\partial_{kn}\left(\frac{\dot{\mathcal{I}}^{jkn} - 2\epsilon^{mjk}\mathcal{J}^{mn}}{r}\right) + \cdots\right],$$

$$h^{jk} = \frac{4G}{c^4}\left[\frac{1}{2}\frac{\ddot{\mathcal{I}}^{jk}}{r} - \frac{1}{6}\partial_n\left(\frac{\ddot{\mathcal{I}}^{jkn} + 2\epsilon^{mnj}\dot{\mathcal{J}}^{mk} + 2\epsilon^{mnk}\dot{\mathcal{J}}^{mj}}{r}\right) + \frac{GM^2}{4r^2}n^j n^k + \cdots\right].$$

The potentials are expressed in terms of $n^j = x^j/r$, and in terms of multipole moments that depend on retarded time $\tau = t - r/c$; overdots indicate differentiation with respect to τ. In h^{00} the mass term contains 0PN and 1PN contributions, the quadrupole term is a 1PN contribution, and the octupole and M^2 terms are 1.5PN contributions. The first two terms in h^{0j} are 1PN contributions, while the rest are 1.5PN. And finally, the quadrupole term in h^{jk} is a 1PN contribution, while the remaining terms are all 1.5PN contributions.

We have the total gravitational mass

$$M = \int \rho^*\left[1 + \frac{1}{c^2}\left(\frac{1}{2}v^2 - \frac{1}{2}U + \Pi\right)\right] d^3x + O(c^{-4}),$$

the total angular momentum

$$\boldsymbol{J} = \int \rho^* \boldsymbol{x}\times\boldsymbol{v}\, d^3x + O(c^{-2}),$$

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and the mass and current multipole moments

$$\mathcal{I}^{jk}(\tau) = \int \rho^* x^j x^k \, d^3x + O(c^{-2}),$$

$$\mathcal{I}^{jkn}(\tau) = \int \rho^* x^j x^k x^n \, d^3x + O(c^{-2}),$$

$$\mathcal{J}^{jk}(\tau) = \epsilon^{jab} \int \rho^* x^a v^b x^k \, d^3x + O(c^{-2}).$$

We recall that M and $\boldsymbol{J}$ are conserved quantities. The gravitational potentials are evaluated in the center-of-mass frame, in which the total momentum $\boldsymbol{P}$ and center-of-mass position $\boldsymbol{R}$ are set equal to zero.

The multipole moments must be differentiated a number of times before they are inserted within the gravitational potentials. These operations are aided by the identity

$$\dot{F} = \int \rho^* \frac{df}{dt} \, d^3x,$$

where $F(t) := \int \rho^*(t, \boldsymbol{x}) f(t, \boldsymbol{x}) \, d^3x$ and $df/dt = \partial_t f + v^j \partial_j f$; this is established on the basis of the continuity equation $\partial_t \rho^* + \partial_j(\rho^* v^j) = 0$, as shown back in Sec. 1.4.3. The terms involving $d\boldsymbol{v}/dt$ are handled by invoking Euler's equation $\rho^*(dv^j/dt) = \rho^* \partial^j U - \partial^j p + O(c^{-2})$, which was shown in Sec. 7.3.2 to be a consequence of energy-momentum conservation.

In the far-away wave zone, where $r \gg \lambda_c$, the gravitational potentials reduce to

$$h^{00} = \frac{4G}{c^2 r}\left[M + \frac{1}{2c^2} \ddot{\mathcal{I}}^{jk} n_j n_k + \frac{1}{6c^3} \dddot{\mathcal{I}}^{jkn} n_j n_k n_n + \cdots \right],$$

$$h^{0j} = \frac{4G}{c^3 r}\left[\frac{1}{2c} \ddot{\mathcal{I}}^{jk} n_k + \frac{1}{12c^3}\left(\dddot{\mathcal{I}}^{jkn} - 2\epsilon^{mjk} \ddot{\mathcal{J}}^{mn} \right) n_k n_n + \cdots \right],$$

$$h^{jk} = \frac{4G}{c^4 r}\left[\frac{1}{2} \ddot{\mathcal{I}}^{jk} + \frac{1}{6c}\left(\dddot{\mathcal{I}}^{jkn} + 2\epsilon^{mnj} \ddot{\mathcal{J}}^{mk} + 2\epsilon^{mnk} \ddot{\mathcal{J}}^{mj} \right) n_n + \cdots \right].$$

The time-dependent piece of $h^{\alpha\beta}$ is dominated by the quadrupole moment of the mass distribution.

7.5 Bibliographical notes

The implementation of post-Minkowskian theory presented in this chapter is based on the DIRE approach (Direct Integration of the Relaxed Einstein equations) of Will and Wiseman (1996) and Pati and Will (2000 and 2001).

The fast-motion implementation of the theory reviewed in Box. 7.6 was attempted by Goldberg, Havas, Rosenblum, and coworkers. Representative papers are Havas and Goldberg (1962), Smith and Havas (1965), and Rosenblum (1978).

7.6 Exercises

7.1 Show that in Eq. (7.13), the second term in the retarded expansion of $h^{00}_{\mathscr{N}}$ is given by the surface integral

$$\delta h^{00}_{\mathscr{N}} = \frac{4G}{c^4} \oint_{\partial \mathscr{M}} \tau^{0j} \, dS_j \, .$$

Using the first term of Eq. (7.48b) to estimate τ^{0j} in the wave zone, and taking the monopole and quadrupole contributions to h^{00} from Box 7.7, show that

$$\delta h^{00}_{\mathscr{N}} \sim \frac{G^2}{c^{10}} \dddot{\mathcal{I}}^{jk} \dddot{\mathcal{I}}^{jk}$$

after discarding terms that depend on the cutoff radius $\mathcal{R}$. Show that this makes a contribution to h^{00} at 4PN order.

7.2 Verify the identities of Eqs. (7.14). Using these, verify that the odd-order terms in Eq. (7.12) take the forms displayed in Eqs. (7.15), modulo surface terms.

7.3 In this problem we prove that at first post-Newtonian order, the integral of Eq. (7.59) defining the total mass M is insensitive to the wave-zone aspects of the integrand. To show this, decompose the integral into a near-zone portion $r < \mathcal{R}$ and a wave zone portion $r > \mathcal{R}$. Show that the $\partial_j(U\partial^j U)$ term in the energy-momentum pseudotensor makes a contribution

$$\Delta M_{\rm near} = \frac{7G}{2c^2} \frac{M^2}{\mathcal{R}}$$

to the near-zone integral. Next, use the expression of Eq. (7.110) to show that the wave-zone contribution to the mass is given by

$$\Delta M_{\rm wave} = -\frac{7G}{2c^2} \frac{M^2}{\mathcal{R}} .$$

Conclude that these contributions cancel out, and that the wave-zone portion of the integral makes no essential contribution to the mass.

7.4 As we saw in Sec. 7.3.3, the Poisson equation $\nabla^2 f = \nabla^2 g$ has the solution

$$f = g - \frac{1}{4\pi} \oint_{\partial \mathscr{M}} \left(\frac{\partial'^j g'}{|\boldsymbol{x} - \boldsymbol{x}'|} - g' \partial'_j \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} \right) dS'_j \, .$$

Show that the surface term satisfies Laplace's equation for any point $\boldsymbol{x}$ within the near zone.

7.5 Consider the superduperpotential of ρ^*, defined by

$$Y(t, \boldsymbol{x}) := G \int \rho^*(t, \boldsymbol{x}') |\boldsymbol{x} - \boldsymbol{x}'|^3 \, d^3x' \, .$$

(a) Show that $\nabla^2 Y = 12X$, where X is the superpotential.

(b) Following the method of Box 7.3, show that the solution to $\nabla^2 Y = 12X$ can be expressed as

$$Y(t, \boldsymbol{x}) = G \int \rho^*(t, \boldsymbol{y}) K(\boldsymbol{x}; \boldsymbol{y})\, d^3y + Y_0(t, \boldsymbol{x}),$$

in terms of a two-point function K that satisfies $\nabla^2 K = 12|\boldsymbol{x} - \boldsymbol{y}|$; Y_0 is a solution to Laplace's equation.

(c) Calculate the two-point function, and determine Y_0 so that your answer for Y agrees with its starting definition.

7.6 Show that the quadrupole-moment piece of the wave-zone potential h^{00} in Box 7.7 is given explicitly by

$$\frac{2G}{c^2}\left(\frac{1}{c^2 r}\ddot{\mathcal{I}}^{jk} + \frac{3}{cr^2}\dot{\mathcal{I}}^{\langle jk\rangle} + \frac{3}{r^3}\mathcal{I}^{\langle jk\rangle}\right) n_j n_k \,.$$

7.7 For a static, spherically-symmetric source, show that the wave-zone potentials given in Box 7.7 reduce to

$$\begin{aligned} h^{00} &= \frac{4GM}{c^2 r} + 7\left(\frac{GM}{c^2 r}\right)^2 + \cdots \,, \\ h^{0j} &= 0 \,, \\ h^{jk} &= \left(\frac{GM}{c^2 r}\right)^2 n^j n^k + \cdots \end{aligned}$$

Verify that this corresponds to the post-Newtonian expansion of the Schwarzschild metric in harmonic coordinates.

7.8 The total mass of a gravitating system is defined by the integral

$$M = \frac{1}{c^2}\int (-g)\big(T^{00} + t^{00}_{\mathrm{LL}}\big)\, d^3x \,.$$

But the mass parameter that appears in the leading-order contribution to h^{00} in the wave zone is

$$M_0 = \frac{1}{c^2}\int_{\mathscr{M}} (-g)\big(T^{00} + t^{00}_{\mathrm{LL}} + t^{00}_{\mathrm{H}}\big)\, d^3x \,.$$

Both masses satisfy a conservation law, because $\partial_\beta[(-g)t^{\alpha\beta}_{\mathrm{H}}] = 0$ identically. This problem explores whether $(-g)t^{00}_{\mathrm{H}}$ makes a contribution to the *value* of the mass.

(a) Defining $\tilde{t}^{\alpha\beta}_{\mathrm{H}} := (16\pi G/c^4)(-g)t^{\alpha\beta}_{\mathrm{H}} = \partial_\mu h^{\alpha\nu}\partial_\nu h^{\beta\mu} - h^{\mu\nu}\partial_{\mu\nu}h^{\alpha\beta}$, and using the harmonic gauge condition $\partial_\beta h^{\alpha\beta} = 0$, show that

$$\begin{aligned} \tilde{t}^{\alpha\beta}_{\mathrm{H}} &= 2\partial_0 h^{\alpha 0}\partial_0 h^{\beta 0} + 2h^{0(\alpha}\partial_0^2 h^{\beta)0} - h^{00}\partial_0^2 h^{\alpha\beta} \\ &\quad - 2\partial_0 h^{00}\partial_0 h^{\alpha\beta} - h^{\alpha\beta}\partial_0^2 h^{00} + \partial_j f^{j\alpha\beta} \,, \end{aligned}$$

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where

$$f^{j\alpha\beta} := 2h^{0(\alpha}\partial_0 h^{\beta)j} + h^{k(\alpha}\partial_k h^{\beta)j} + h^{j(\alpha}\partial_0 h^{\beta)0} - 2h^{0j}\partial_0 h^{\alpha\beta} - h^{jk}\partial_k h^{\alpha\beta} - \partial_0 h^{0j} h^{\alpha\beta} .$$

(b) Using this expression, show that the contribution of the harmonic energy-momentum pseudotensor to a near-zone momentum

$$P_0^\alpha := \frac{1}{c}\int_{\mathscr{M}} \tau^{\alpha 0}\, d^3x$$

and a near-zone angular momentum

$$J_0^{\alpha\beta} := \frac{2}{c}\int_{\mathscr{M}} x^{[\alpha}\tau^{\beta]0}\, d^3x$$

comes from integrals over the surface bounding the domain of integration.

(c) Show that $f^{j00} = \partial_k(h^{0j}h^{0k} - h^{00}h^{jk})$.

(d) Using the wave-zone form of the potentials from Box 7.7, and keeping only terms that are independent of the cutoff radius $\mathcal{R}$, show that M and M_0 are related by

$$M = M_0 - \frac{2}{3}\frac{GM_0}{c^5}\overset{\cdots}{\mathcal{I}}{}^{kk}(\tau) + O(c^{-7}) .$$

Show that the second term is a correction of order $(v_c/c)^5$ relative to the first term.

7.9 This problem explores how to solve the Landau–Lifshitz formulation of the Einstein field equations for the Schwarzschild geometry.

(a) Assuming static spherical symmetry, show that the general form of the gothic inverse metric in Cartesian coordinates can be written in the form

$$\mathfrak{g}^{00} = N(r) ,$$
$$\mathfrak{g}^{0j} = 0 ,$$
$$\mathfrak{g}^{jk} = \alpha(r)P^{jk} + \beta(r)n^j n^k ,$$

where N, α and β are arbitrary functions of r, n^j is a radial unit vector, and $P^{jk} := \delta^{jk} - n^j n^k$.

(b) Show that $\mathfrak{g}_{\alpha\beta}$ is given by $\mathfrak{g}_{00} = N^{-1}$, $\mathfrak{g}_{jk} = \alpha^{-1}P^{jk} + \beta^{-1}n^j n^k$, and that $\mathfrak{g} := \det[\mathfrak{g}^{\alpha\beta}] = N\alpha^2\beta$.

(c) Show that the imposition of the harmonic gauge condition leads to the constraint

$$\beta' = \frac{2}{r}(\alpha - \beta) ,$$

where a prime indicates differentiation with respect to r. Recall that $\partial^j F(r) = F'(r)n^j$, and $\partial^j n^k = r^{-1}P^{jk}$.

(d) Show that the three field equations that arise from the vacuum wave equation $\Box \mathfrak{g}^{\alpha\beta} = (16\pi G/c^4)\tau^{\alpha\beta}$ in harmonic coordinates have the form

$$X' + XY + \frac{1}{r}(2X - Y) = Q\,,$$
$$XY + \frac{1}{r}(2X + Y) = -Q\,,$$
$$Z' + YZ + \frac{2}{r}Z = Q\,,$$

where

$$X := \frac{\alpha'}{\alpha}\,, \quad Y := \frac{\beta'}{\beta}\,, \quad Z := \frac{N'}{N}\,,$$

and

$$Q := \frac{1}{8}\Big(3Y^2 - Z^2 + 2YZ + 4XZ - 4XY\Big)\,.$$

Hint: One equation comes from the 00 component of the field equations, the other two come from splitting the jk components into a piece proportional to $n^j n^k$ and another piece proportional to P^{jk}. Use the gauge condition to simplify your expressions.

(e) By combining the first two field equations, obtain the solutions

$$X = 0 \quad \text{or} \quad r^4\beta^2 X = c\,,$$

where $c \neq 0$ is a constant.

(f) Choosing the solution $X = 0$, show that the solutions for α and β that satisfy appropriate asymptotic conditions at $r = \infty$ are

$$\alpha = 1\,, \quad \beta = 1 - \frac{a}{r^2}\,,$$

where a is an arbitrary constant. Find the solution for N, determine a, and verify that the result is the Schwarzschild metric in harmonic coordinates.

(g) What is your interpretation of the second class of solutions, represented by a non-zero value of c? Show that by combining the equation $r^4\beta^2 X = c$ with the gauge condition, you can eliminate α and obtain the following differential equation for β:

$$W'' - \frac{W'}{r} = c\frac{W'}{W^2}\,,$$

where $W := r^2\beta$. Spend some time (but not too much!) trying to find a closed form solution to this non-linear equation. (If you find one, please send it to us!)

7.10 Consider the harmonic gauge condition of Eq. (5.175), $\Box_g X^{(\mu)} = 0$, which is a scalar wave equation for the four scalar fields T, X, Y and Z. Using the metric in Schwarzschild coordinates to calculate the operator $\Box_g$, and defining $T := t$, $X := r_{\rm h}(r)\sin\theta\cos\phi$, $Y := r_{\rm h}(r)\sin\theta\sin\phi$, and $Z := r_{\rm h}(r)\cos\theta$, show that the harmonic condition reduces to a single differential equation for $r_{\rm h}(r)$, a Legendre equation

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of degree $\ell = 1$. Show that the solution that satisfies the condition that $r_{\rm h} \to r$ as $r \to \infty$ is given by

$$r_{\rm h} = r - \frac{1}{2}R + b\left[\left(r - \frac{1}{2}R\right)\ln\left(1 - \frac{R}{r}\right) + R\right],$$

where $R = 2GM/c^2$ and b is an arbitrary constant. What do you conclude about the uniqueness of harmonic coordinates? (We encounter this question again in Sec. 11.1.5, in the context of gravitational waves.) Is there a link between this and the second class of solutions in part (g) of the previous problem?